In this lecture we will study about constant depth circuits.

**Definition 1.** A constant depth circuit is a circuit with a constant depth $d$ and computes a function $f : \sum^n \rightarrow \sum$.

In the definition of circuit, we require that “AND” and “OR” gates have fan-in two. Thus if a circuit with depth $d$ computes a function, the function can depend on at most $2^d$ bits. Thus any function, for example Parity, that depends on all its input bits can not be computed by such circuits.

We now relax the restriction on the fan-in. For rest of the lecture we study constant depth circuits with unbounded fan-in.

**Definition 2.** A boolean function $f : \sum^* \rightarrow \sum$ is in $AC_0$ if there is a constant $d$, a polynomial $p$ and a family of circuits $(C_1, C_2, \ldots)$ such that

1. $C_n$ computes $f : \sum^n \rightarrow \sum$
2. For all $n$, $\text{depth}(C_n) \leq d$
3. For all $n$, $\text{size}(C_n) \leq p(n)$

If the fan-in is bounded the class is called $NC_0$.

If there is a negation gate in the circuit, it can be pushed to the bottom most level using de-morgan laws. Thus we can think of $AC_0$ circuit having input gates labelled with $x_i$ or $\overline{x_i}$ and all internal gates are either “AND” or “OR” gate. Thus we assume that there are no negation gates.

**Definition 3.** A layered formula is a circuit whose underlying graph is a tree and satisfies the following conditions.

1. Out degree of each gate is 1
2. In-degree of each gate is unbounded
3. At any layer $d$, all gates are either $\lor$ or $\land$.
4. If layer $i$ contains $\lor$ gates, then layer $i + 1$ contains $\land$ gates.
5. If layer $i$ contains $\land$ gates, then layer $i + 1$ contains $\lor$ gates.
6. Wires connect from layer $i$ to layer $i + 1$ only

The nodes at layer 0 are the leaf nodes.

**Observations:**

1. Every circuit $C$ with size $s$ and depth $d$ can be converted into a layered formula of size $O(s^d)$ and depth $2d$.
2. A function $f$ belongs to $AC_0$ if and only if it has a poly size layered formula for every input size $n$
3. Every layered formula of depth 2 is either a DNF or a CNF. Since every function has exponential-size CNF/DNF, every function has exponential-size depth two layered formula.

**Lemma 1.** Let $\phi$ be a DNF formula for $\oplus_n$ where $\phi = T_1 \lor T_2 \lor T_3 \cdots T_l$. Then, each $T_i$ must have atleast $n$ literals and $l \geq 2^{n-1}$
Proof. Suppose there is a term $T_i$ that has less than $n$ literals. This means that there exists a variable $x_i$ where $T_i$ contains neither $x_i$ nor $\bar{x}_i$. This in turn implies that the parity function does not depend on the variable $x_i$. But that is not possible. Parity function depends on the values of all the $n$ variables.

For $n$ input bits the number of assignments for which the parity is equal to 1 is $2^{n-1}$. Each term in the DNF is 1 for exactly one setting of input variables. So, there should be at least $2^{n-1}$ terms in $\phi$.

**Corollary 1.** Every depth 2 layered formula for $\oplus_n$ must have a size $\geq 2^{n-1}$ and fan-in for every bottom level gate is greater than or equal to $n$.

**Lemma 2.** There is a $O(n2^{n^{1/4}})$ size, depth $d$ layered formula for $\oplus_n$.

**Proof.** We use a divide and conquer approach to build our layered formula. First stage of the construction works as follows: Divide the $n$ bits into block of $n^{1/d}$ bits each. For each block compute parity using a a layered formula of size $2^{n^{1/4}}$ and depth 2. This gives $n^{1-1/d}$ bits. Parity of the original string equals the parity of these $n^{1-1/d}$ bits. We can determine the parity of these bits in the same way using recursion. This will result in a circuit of size $O(n2^{n^{1/4}})$ and depth $2d$. We can reduce the depth to $d$ as follows. To compute a parity of a parity we can use either a DNF layered formula or a CNF layered formula. We will alternate between DNF and CNF formulas. If we use DNF formula during stage 1, then we use CNF formula during stage two. Top gate of DNF formula is and “OR” and bottom gate of a CNF formula are “OR”. So we we can merge them into one layer. This reduces depth to $d$. □

Given a circuit with $n$ variables, let $R_t$ be the set of all partial assignments that assigns values to $n - t$ variables. Let $a \in R_t$. If $f : \sum^n \rightarrow \sum$ then $f|_a : \sum^t \rightarrow \sum$. Every term in a $k$-DNF has atmost $k$ literals and every term in a $k$-CNF has atmost $k$ literals.

**Lemma 3.** [Hastad’s Switching Lemma] Let $f$ be a $k$-DNF formula. Then, for every $t \leq \sqrt{n}$ and for all $s > 2$, $Pr_{a \in R_t}[f|_a$ cannot be expressed as $s$-CNF] $\leq \left(\frac{10b}{\sqrt{n}}\right)^{\frac{t}{2}}$

We can prove that Parity does not have small size constant depth circuits using the switching lemma.

**Theorem 1.** Parity cannot be computed by constant depth poly size layered formulas. (Parity $\notin AC_0$)

**Proof.** We make use of random restriction and Hastads switching lemma to prove this theorem.

Assume that $\oplus_n$ has a layered formula of size $n^b$ and depth $d$. Let $n_i = \sqrt{n_{i-1}}$ and $k_i = 2^{i}10b$. Set $n_0 = n$. Without loss of generality assume that the bottom level gates are $\wedge$ and the bottom fan-in is $k_0$. Then, the bottom two levels will contain many DNF formulas. Fix a DNF formula $g$.

Let $a \in R_{\sqrt{n}}$. $n_0 = n$ and $k_0 = 10b$.

$$Pr_{a \in R_{\sqrt{n}}}[g|_a$ is not a $k_1$-CNF] \leq \left(\frac{10b}{\sqrt{n}}\right)^{10b} \leq \frac{1}{2n^b}$$
In the bottom 2 levels there can be more than one DNF formula. Lets denote them $g^1, g^2 \cdots g^m$. The probability that each of those DNF’s can be expressed as a $k_1$-CNF is given as follows. There can be atmost $n^b$ DNF’s in the bottom two levels.

\[
Pr_{a \in R_{\sqrt{n}}}[\text{Output at every } g^i_a \text{ is a } k_1\text{-CNF}]
\geq 1 - \sum_{i=m}^{1} Pr_{a \in R_{\sqrt{n}}}[g^i_a \text{ is not a } k_1\text{-CNF}]
\geq 1 - n^b \times \frac{1}{2n^b}
\geq \frac{1}{2}
\]

This implies that there is a partial assignment $a$ such that the output of all gates at level 2 can be represented as $k_1$-CNF. Fix such an assignment. Since the circuit $C$ is layered formula, we can reduce the depth by one. Thus $C|_a$ is a circuit of depth $d - 1$, with bottom level fan-in at most $k_1$ and is computing the parity (or its negation) of $\sqrt{n}$ bits.

Now applying the same argument as above another $d - 3$ times would result in a circuit of depth 2, bottom level fan-in $k_{d-2}$. This circuit is computing parity (or its negation) of $n_{d-2} = n^{2^{d-2}}$ bits. Since $k_{d-2} = 10b2^{d-2}$. For large enough $n$ the value $k_{d-2}$ is less than $n_{d-2}$. This contradicts Corollary 1.

□