

# Strong Bisimulation

## Overview

- Actions
- Labeled transition system
- Transition semantics
- Simulation
- Bisimulation

## References

- Robin Milner, "Communication and Concurrency"
- Robin Milner, "Communicating and Mobil Systems"

# Actions and States

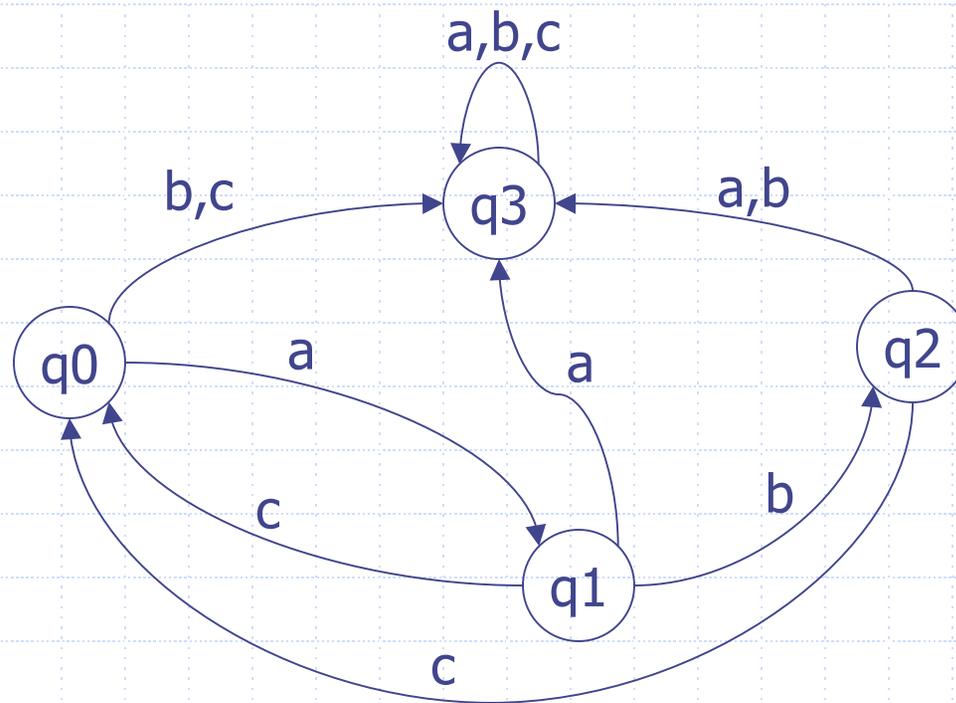
## Actions:

- We presuppose an infinite set  $N$  of names; we use  $a, b, \dots$  to range over  $N$ .
- Then we introduce the set  $\overline{N} = \{\overline{a} \mid a \in N\}$ , which we call co-names.
- We assume that  $N$  and  $\overline{N}$  are disjoint, and we denote their union  $N \cup \overline{N}$ , by  $L$ , the set of labels (the kind of labels, which identifies the buttons on our black boxes). For the moment, the set  $L$  and  $\Sigma$  coincide.

## Conceptual changes:

- What matters about a string  $s$  - a sequence of actions - is not whether it drives the automaton into an accepting state (since we cannot detect this by interaction) but whether the automaton is able to perform the sequence of  $s$  interactively.
- A labeled transition system can be thought of as an automaton without a start or accepting states.
- *Any* state can be considered as the start.

# General Automaton



# Labeled Transition System

A labeled transition system over actions  $\Sigma$  is a pair  $(Q, T)$  consisting of:

- a set  $Q = \{q_0, q_1, \dots\}$  of *states*,
- a ternary relation  $T \subseteq (Q \times \Sigma \times Q)$ , known as a *transition relation*.

If  $(q, a, q') \in T$  we write  $q \xrightarrow{a} q'$ , and we call  $q$  the source and  $q'$  the target of the transition.

Alternative definition:  $(S, T, \{\xrightarrow{t} : t \in T\})$

- $S$  is a set of states
- $T$  is a set of transition labels
- $\xrightarrow{t} \subseteq S \times S$  is a transition relation for each  $t \in T$ .

# LTS and Automaton

- ◆ An LTS can be thought of as an automaton without a start or accepting states.
- ◆ By omitting the start state, we gain the freedom to consider any state as the start.
- ◆ Each selection of a start defines a different automaton, but is based upon the same LTS.

# Strong Simulation - Idea

- ◆ In 1981 D. Park proposed a new approach to define the equivalence of automata - bisimulation.
- ◆ Given a labeled transition system there exists a standard definition of bisimulation equivalence that can be applied to this labeled transition system.
- ◆ The definition of bisimulation is given in a *coinductive* style that is, two systems are bisimilar if we cannot show that they are not.
- ◆ Informally, to say a "system S1 simulates system S2" means that S1's observable behavior is at least as rich as that of S2.

# Strong Simulation - Definition

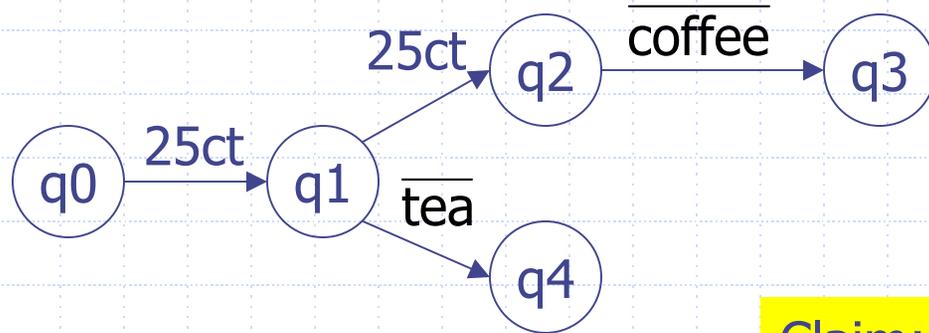
Let  $(Q, T)$  be an labeled transition system, and let  $\mathbf{S}$  be a binary relation over  $Q$ . Then  $\mathbf{S}$  is called a strong simulation over  $(Q, T)$  if, whenever  $p \mathbf{S} q$ ,

If  $p \xrightarrow{a} p'$  then there exists  $q' \in Q$  such that  $q \xrightarrow{a} q'$  and  $p' \mathbf{S} q'$ .

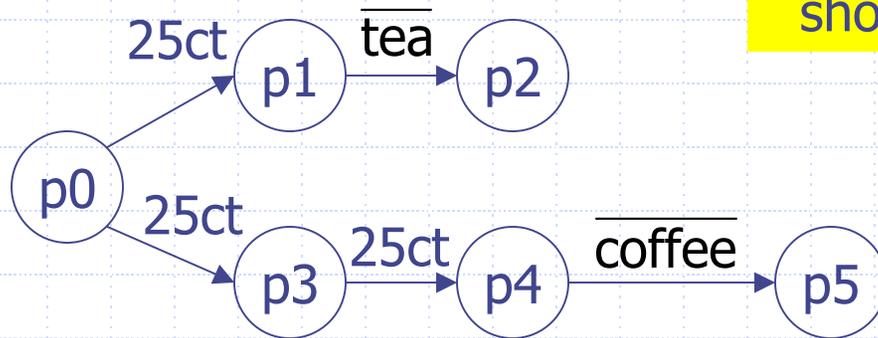
We say that  $q$  strongly simulates  $p$  if there exists a strong simulation  $\mathbf{S}$  such that  $p \mathbf{S} q$ .

# Strong Simulation - Example

S1:



S2:



Claim:

The states q0 and p0 are different.  
Therefore, the systems S1 and S2  
should not be considered equivalent.

# Defining $\mathbf{S}$

If we define

$$\mathbf{S} = \{(p_0, q_0), (p_1, q_1), (p_3, q_1), (p_2, q_4), (p_4, q_2), (p_5, q_3)\}$$

then  $\mathbf{S}$  is a strong simulation; hence  $S_1$  strongly simulates  $S_2$ .

- ◆ To verify this, for every pair  $(p, q) \in \mathbf{S}$  we have to consider each transition of  $p$ , and show that it is properly matched by some transition of  $q$ .
- ◆ However, there exists no strong simulation  $\mathbf{R}$  that contains the pair  $(q_1, p_1)$ , because one of  $q_1$ 's transition could never be matched by  $p_1$ . Therefore, the states  $q_0$  and  $p_0$  are different, and the systems  $S_1$  and  $S_2$  are not considered to be equivalent.

# Strong Bisimulation

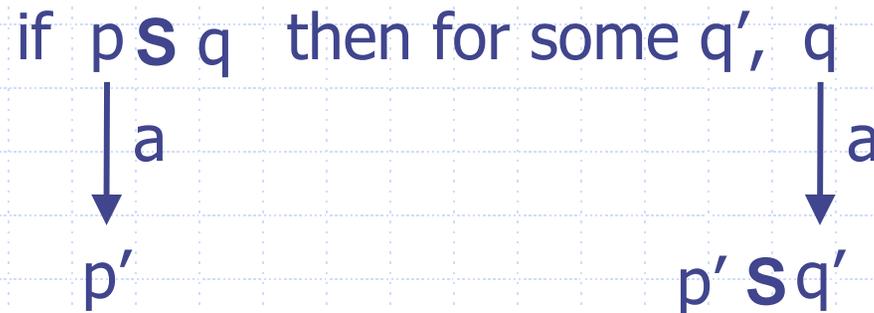
The converse  $\mathbf{R}^{-1}$  of any binary relation  $\mathbf{R}$  is the set of pairs  $(y, x)$  such that  $(x, y) \in \mathbf{R}$ .

Let  $(Q, \mathcal{T})$  be an labeled transition system, and let  $\mathbf{S}$  be a binary relation over  $Q$ . Then  $\mathbf{S}$  is called a strong bisimulation over  $(Q, \mathcal{T})$  if both  $\mathbf{S}$  and its converse  $\mathbf{S}^{-1}$  are strong simulations.

We say that that the states  $p$  and  $q$  are strongly bisimilar or strongly equivalent, written  $p \sim q$ , if there exists a strong bisimulation  $\mathbf{S}$  such that  $p \mathbf{S} q$ .

# Diagrams

The condition for  $\mathbf{S}$  to be a strong bisimulation can be expressed in diagrams:



Thus  $q'$  strongly simulates  $p'$ , or  $p'$  is strongly simulated by  $q'$ , means that whatever transition path  $p$  takes,  $q$  can match it by a path, which retains all of  $p$ 's options.

# Bisimulation - a Board Game

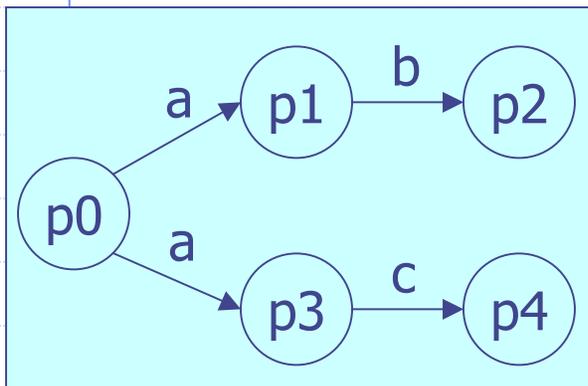
- ◆ Checking the equivalence of interactive systems can be considered a board game between two persons, the “unbeliever”, who thinks that  $S1$  and  $S2$  are not equivalent, and the “believer”, who thinks that  $S1$  and  $S2$  are equivalent.
- ◆ The underlying strategy of this game is that the (demonic) unbeliever is trying to perform transitions, which the cannot be matched by the (angelic) believer. The unbeliever loses if there are no transitions left for either systems, whereas the believer loses, if he cannot match a move made by the unbeliever.

# Working With Simulations

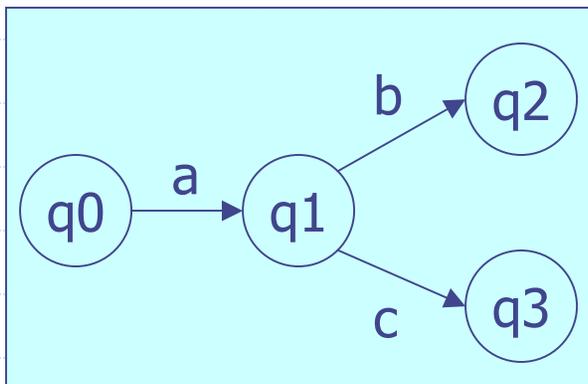
- ◆ What do we do with (bi)simulations?
  - Exhibiting a (bi)simulation:  
“guessing” a relation  $\mathbf{S}$  that contains  $(p,q)$
  - Checking a (bi)simulation:  
checking that a given relation  $\mathbf{S}$  is in fact a (bi)simulation
- ◆ There exist algorithms and tools (e.g. CWB) that can generate relations that – by construction – satisfy the property of being a (bi)simulation.
- ◆ Results on (semi-)decidability are very important for such tools.

# Checking Bisimulation

S1:



S2:



$S1 \sim S2?$

To construct  $\mathbf{S}$  start with  $(p_0, q_0)$  and check whether S2 can match all transitions of S1:

$$\mathbf{S} = \{ (p_0, q_0), (p_1, q_1), (p_3, q_1), (p_2, q_2), (p_4, q_3) \}$$

System S2 can simulate system S1. Now check, whether  $\mathbf{S}^{-1}$  is a simulation or not:

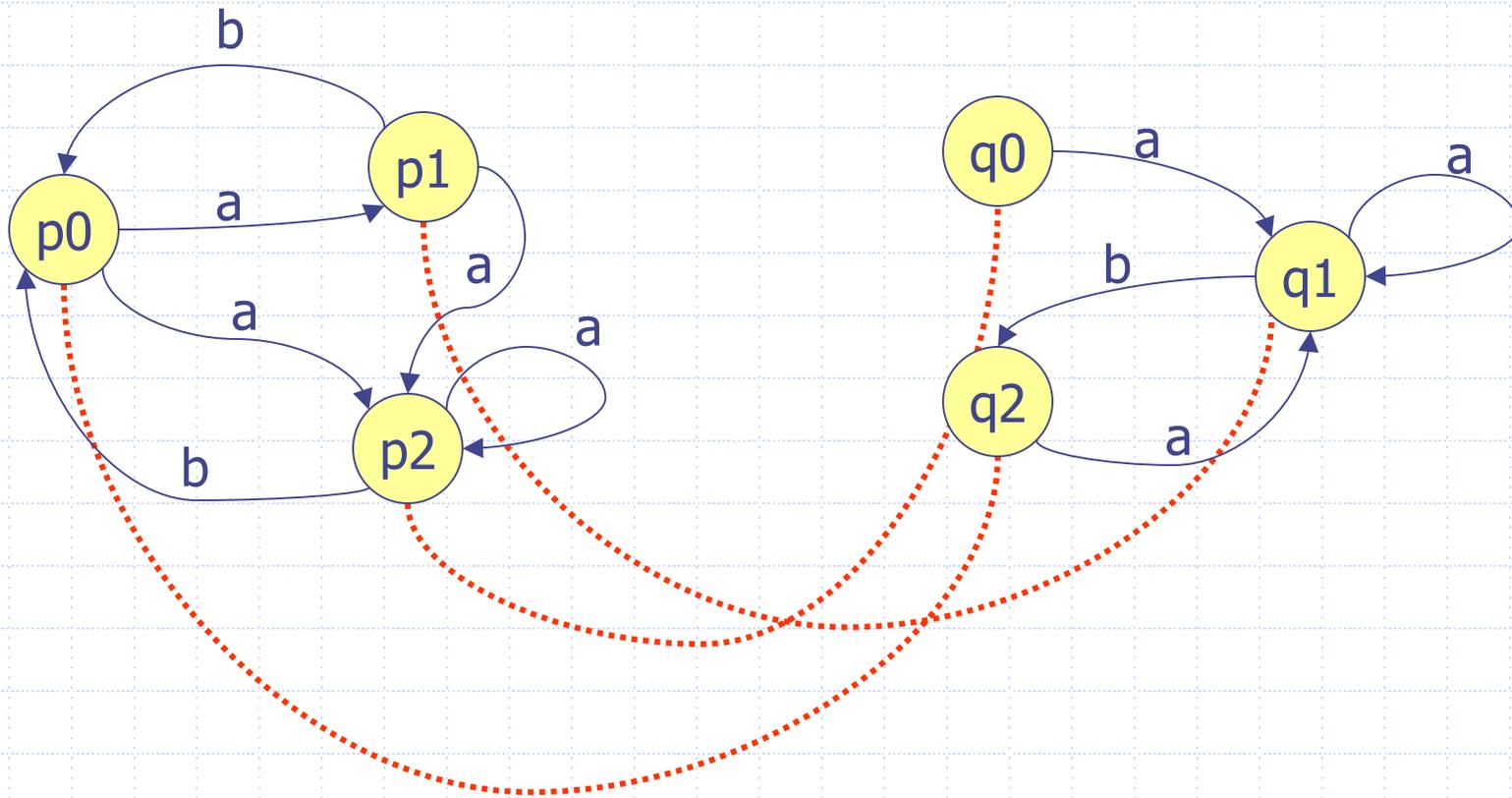
$$\mathbf{S}^{-1} = \{ (q_0, p_0), (q_1, p_1), (q_1, p_3), (q_2, p_2), (q_3, p_4) \}$$

Start with  $(q_0, p_0) \in \mathbf{S}^{-1}$ .

- 1:  $q_0$  has one transition 'a' that can be matched by two transitions of S1 (target  $p_1$  and  $p_3$ , respectively) and we have  $(q_1, p_1) \in \mathbf{S}^{-1}$  and  $(q_1, p_3) \in \mathbf{S}^{-1}$ .
- 2:  $q_1$  has two transitions 'b' and 'c', which, however, cannot be appropriately matched by the related states  $p_1$  and  $p_3$  of system S1 ( $p_1$  has only a 'b' transition whilst  $p_3$  has only a 'c' transition).

We have, therefore,  $S1 \not\sim S2$ .

# Linking States



# $\sim$ is an Equivalence Relation

◆  $p \sim p$

◆  $p \sim q$  implies  $q \sim p$

◆  $p \sim q$  and  $q \sim r$  imply  $p \sim r$

# Reflexivity

- ◆ Let  $Q$  be a process and  $Id_Q = \{(p,p) \mid p \in Q\}$ . For reflexivity, it is enough to show that  $Id_Q$  is a bisimulation.

Proof:

Suppose  $Id_Q = \{(p,p) \mid p \in Q\}$ . We have to show that for all  $(p,p) \in Id_Q$ , if  $p \xrightarrow{a} p'$ , then there exists  $q'$  such that  $p \xrightarrow{a} q'$  and  $(p',q') \in Id_Q$ . Now, let  $p \xrightarrow{a} p'$ , if  $p \xrightarrow{a} p'$ , then we have to find a state  $q' \in Q$  such that  $p \xrightarrow{a} q'$  and  $p' Id_Q q'$ . By assumption,  $p' \in Q$ , we take  $q' = p'$ , hence  $p \xrightarrow{a} p'$ , and by definition of  $Id_Q$ , we have  $p' Id_Q p'$ , as required.

Finally, since  $Id_Q = Id_Q^{-1}$ ,  $Id_Q$  is a bisimulation.

q.e.d.

# Symmetry

For symmetry, we have to show that if  $\mathbf{S}$  is a bisimulation then so is its converse  $\mathbf{S}^{-1}$ . However, this is obvious from the definition of bisimulation.

# Transitivity

$$\mathbf{S}_1 \circ \mathbf{S}_2 = \{(p, r) \mid q \text{ exists with } (p, q) \in \mathbf{S}_1 \text{ and } (q, r) \in \mathbf{S}_2\}$$

Proof:

Let  $(p, r) \in \mathbf{S}_1 \circ \mathbf{S}_2$ . Then there exists a  $q$  with  $(p, q) \in \mathbf{S}_1$  and  $(q, r) \in \mathbf{S}_2$ .

( $\rightarrow$ ) If  $p \xrightarrow{a} p'$ , then since  $(p, q) \in \mathbf{S}_1$  there exists  $q'$  and  $q \xrightarrow{a} q'$  and  $(p', q') \in \mathbf{S}_1$ . Furthermore, since  $(q, r) \in \mathbf{S}_2$  there exists a  $r'$  with  $r \xrightarrow{a} r'$  and  $(q', r') \in \mathbf{S}_2$ . Due to the definition of  $\mathbf{S}_1 \circ \mathbf{S}_2$  it holds that  $(p', r') \in \mathbf{S}_1 \circ \mathbf{S}_2$  as required.

( $\leftarrow$ ) similar to ( $\rightarrow$ ).

# Fact

$\sim$  is the largest strong bisimulation, that is,  $\sim$  is a strong bisimulation and includes any other such.

Assume that each  $\mathbf{S}_i$  ( $i=1,2,\dots$ ) is a strong bisimulation. Then  $\bigcup_{i \in I} \mathbf{S}_i$  is a strong bisimulation.

Let each  $\mathbf{S}_i$  ( $i=1,2,\dots$ ) be a strong bisimulation. We have to show that  $\bigcup_{i \in I} \mathbf{S}_i$  is a strong bisimulation.

Let  $(p,q) \in \bigcup_{i \in I} \mathbf{S}_i$ . If  $p \xrightarrow{a} p'$ , then since  $(p,q) \in \mathbf{S}_i$ ,  $1 \leq i \leq n$ , there exists a  $q' \in \mathbf{S}_i$  with  $q \xrightarrow{a} q'$  and  $(p',q') \in \mathbf{S}_i$  and  $(p',q') \in \bigcup_{i \in I} \mathbf{S}_i$ . By symmetry, the converse holds as well.

# Bisimulation - Summary

- ◆ Bisimulation is an **equivalence relation** defined over a labeled transition system, which respects non-determinism.
- ◆ The bisimulation technique can be used to compare the observable behavior of interacting systems.
- ◆ Note: Strong bisimulation does not cover unobservable behavior, which is present in systems that have operators to define reaction (that is, internal actions).