1. Recall that given any set $C$ and a boundary point $x \in \partial C$, the normal cone at $x$ is defined as $N_C(x) \triangleq \{g : g^\top(y - x) \leq 0, \forall y \in C\}$. Show that $N_C(x)$ is convex.

2. Let $S$ be a convex set in $\mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$ be a square matrix, and $\alpha \in \mathbb{R}$ be a scalar. Show that the following two sets are convex:
   a) $AS = \{y : y = Ax, x \in S\}$
   b) $\alpha S = \{\alpha x : x \in S\}$

3. Let $H(\cdot)$ denote the convex hull operation. Let $S_1$ and $S_2$ be nonempty sets in $\mathbb{R}^n$. Show that $H(S_1 \cap S_2) \subseteq H(S_1) \cap H(S_2)$. Further, is $H(S_1 \cap S_2) = H(S_1) \cap H(S_2)$ true in general? If yes, prove this result; if not, give a counterexample.

4. a) Let $g : \mathbb{R}^n \to \mathbb{R}$ be a concave function, and let $f$ be defined by $f(x) = 1/g(x)$. Show that $f$ is convex over $S = \{x : g(x) > 0\}$. b) State a symmetric result by interchanging the convex and concave functions, and then give a proof.

5. Let $h : \mathbb{R}^n \to \mathbb{R}$ be a convex function, and let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing convex function. Consider the composite function $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) \triangleq g(h(x))$. Show that $f$ is convex.

6. Consider the function $\Theta(u)$ defined by the following optimization problem, where $X$ is a compact polyhedral set:

$$
\Theta(u) \triangleq \text{Minimize } \begin{align*}
    c^\top x + u^\top (Ax - b) \\
    \text{subject to } x \in X
\end{align*}
$$

Show that $\Theta(u)$ is concave in $u$. 