Straight Sliding Trajectory of a Ball

(Supplementary Material for Planning the Initial Motion of a Free Sliding/Rolling Ball, IEEE Transactions on Robotics, 32(3):566–582, 2016)

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This manuscript analyzes the special case where the ball’s initial velocity $v_0$ and angular velocity $\omega_0$, all in the supporting plane under Assumptions (A1) and (A2), are orthogonal to each other. Under Proposition 1, the ball will move along a straight trajectory. It is more convenient to use two orthogonal directions $\hat{i}$ and $\hat{j}$ such that $\hat{i} \times \hat{j} = \hat{z}$. If $v_0 \neq 0$, chose $\hat{i}$ to be in the direction of $v_0$; otherwise, chose $\hat{j}$ to be in the direction of $\omega_0$. For convenience, we write $\omega_0$ for the signed magnitude of $\omega_0$ along $\hat{j}$, and $v_0, v_r, x_r, x_e$ respectively as the signed magnitudes of $v_0, v_r, x_r, x_e$ along $\hat{i}$. Note that $v_0 \geq 0$. The initial sliding velocity by (1) becomes

$$s_0 = u_0 \hat{i} = (v_0 - \rho \omega_0)\hat{i}. \quad (53)$$

If $v_0 < \rho \omega_0$, $s_0$ opposes the initial velocity which consequently increase under sliding friction.

Recall that $t_r$ and $t_e$ are the times at which sliding and rolling ends, respectively. Equations (13), (14), (16), (21), and (23) have become:

$$t_r = \frac{2|v_0 - \rho \omega_0|}{7\mu_s g}, \quad (54)$$
$$v_r = \frac{1}{7}(5v_0 + 2\rho \omega_0), \quad (55)$$
$$x_r = \frac{2}{49\mu_s g}|v_0 - \rho \omega_0|(6v_0 + \rho \omega_0), \quad (56)$$
$$t_e = t_r + \frac{1}{5\mu_r g}|5v_0 + 2\rho \omega_0|, \quad (57)$$
$$x_e - x_r = \frac{1}{10\mu_r g}|5v_0 + 2\rho \omega_0|(5v_0 + 2\rho \omega_0). \quad (58)$$

Suppose that the ball’s velocity $v$ becomes zero at time $t_m$ and location $x_m = x_m \hat{i}$. If this happens during sliding, then we infer from (9) that $v_0$ and $s$ are in the same direction, namely, $v_0 \geq \rho \omega_0$ by (53). In this case, from (9) and (12) we obtain

$$t_m = \frac{v_0}{\mu_s g}, \quad (59)$$
$$x_m = \frac{v_0^2}{2\mu_s g}. \quad (60)$$

The rest of the analysis follows from a case-based reasoning that measures $\rho \omega_0$ in terms of $v_0$. The five scenarios are illustrated in Figure 19.

(1) $\rho \omega_0 < -\frac{v_0}{2}$. Hence $\rho \omega_0 < v_0$. This implies $0 < t_m < t_r$ by (54) and (59). From (53), $v_0$ and $s_0$ are in the same direction. The ball will reverse its motion at $x_m = v_0^2/2\mu_s g$ before sliding ends. Therefore, $x_r < x_m$. We also have $x_e < x_r$ by (58).

(2) $\rho \omega_0 = -\frac{v_0}{2}$. The ball has zero velocity when sliding ends, and thus zero angular velocity by (17). There is no pure rolling phase. We have $x_r = x_e = x_m$ and $t_r = t_e = t_m = v_0/\mu_s g$.

(3) $-\frac{v_0}{2} < \rho \omega_0 < 0$. So the initial sliding velocity $v_0 > 0$. In this case, $t_m > t_r$; that is, the ball’s velocity reduces to zero after sliding ends. This implies $t_e = t_m$. The frictional force increases $\omega$ during sliding.

(4) $\rho \omega_0 = 0$. In this case, $t_r = 0$. The ball starts pure rolling right away.

(5) $\rho \omega_0 > 0$. Here $v_0$ and $s_0$ are in opposite directions. The ball’s velocity increases while angular velocity decreases until $v = \rho \omega$ when pure rolling starts.
Fig. 19. Five scenarios (1)–(5) of a straight ball motion based on $\rho\omega_0$ relative to $v_0$, shown in the upper left corner.