Straight Sliding Trajectory of a Ball

(Supplementary Material for Planning the Initial Motion of a Free Sliding/Rolling Ball)

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This manuscript analyzes the special case where the ball’s initial velocity \( v_0 \) and angular velocity \( \omega_0 \), all in the supporting plane under Assumptions (A1) and (A2), are orthogonal to each other. Under Proposition 1, the ball will move along a straight trajectory. It is more convenient to use two orthogonal directions \( \hat{i} \) and \( \hat{j} \) such that \( \hat{i} \times \hat{j} = \hat{z} \). If \( v_0 \neq 0 \), chose \( \hat{i} \) to be in the direction of \( v_0 \); otherwise, chose \( \hat{j} \) to be in the direction of \( \omega_0 \). For convenience, we write \( \omega_0 \) for the signed magnitude of \( \omega_0 \) along \( \hat{j} \), and \( v_0, v_r, x_r, x_e \), respectively as the signed magnitudes of \( v_0, v_r, x_r, x_e \) along \( \hat{i} \). Note that \( v_0 \geq 0 \).

The initial sliding velocity by (1) becomes

\[ s_0 = u_0 \hat{i} = (v_0 - \rho \omega_0) \hat{i}. \]  

(53)

If \( v_0 < \rho \omega_0 \), \( s_0 \) opposes the initial velocity which consequently increase under sliding friction.

Recall that \( t_r \) and \( t_e \) are the times at which sliding and rolling ends, respectively. Equations (13), (14), (16), (21), and (23) have become:

\[ t_r = \frac{2|v_0 - \rho \omega_0|}{7\mu s g}, \]  

(54)

\[ v_r = \frac{1}{7}(5v_0 + 2\rho \omega_0), \]  

(55)

\[ x_r = \frac{2}{49\mu s g}|v_0 - \rho \omega_0|(6v_0 + \rho \omega_0), \]  

(56)

\[ t_e = t_r + \frac{1}{35\mu s g}|5v_0 + 2\rho \omega_0|, \]  

(57)

\[ x_e - x_r = \frac{1}{35\mu s g}|5v_0 + 2\rho \omega_0|(5v_0 + 2\rho \omega_0). \]  

(58)

Suppose that the ball’s velocity \( v \) becomes zero at time \( t_m \) and location \( x_m = x_m \hat{i} \). If this happens during sliding, then we infer from (9) that \( v_0 \) and \( \hat{s} \) are in the same direction, namely, \( v_0 \geq \rho \omega_0 \) by (53). In this case, from (9) and (12) we obtain

\[ t_m = \frac{v_0}{\mu s g}, \]  

(59)

\[ x_m = \frac{v_0^2}{2\mu s g}. \]  

(60)

The rest of the analysis follows from a case-based reasoning that measures \( \rho \omega_0 \) in terms of \( v_0 \). The five scenarios are illustrated in Figure 19.

1. \( \rho \omega_0 < -\frac{5}{7}v_0 \). Hence \( \rho \omega_0 < v_0 \). This implies \( 0 < t_m < t_r \) by (54) and (59). From (53), \( v_0 \) and \( s_0 \) are in the same direction. The ball will reverse its motion at \( x_m = v_0^2/2\mu s g \) before sliding ends. Therefore, \( x_r < x_m \). We also have \( x_r < x_r \) by (58).

2. \( \rho \omega_0 = -\frac{5}{7}v_0 \). The ball has zero velocity when sliding ends, and thus zero angular velocity by (17). There is no pure rolling phase. We have \( x_r = x_e = x_m \) and \( t_r = t_e = t_m = v_0/\mu s g \).

3. \(-\frac{5}{7}v_0 < \rho \omega_0 < v_0 \). So the initial sliding velocity \( u_0 > 0 \). In this case, \( t_m > t_r \); that is, the ball’s velocity reduces to zero after sliding ends. This implies \( t_e = t_m \). The frictional force increases \( \omega \) during sliding.

4. \( \rho \omega_0 = v_0 \). In this case, \( t_r = 0 \). The ball starts pure rolling right away.

5. \( \rho \omega_0 > v_0 \). Here \( v_0 \) and \( s_0 \) are in opposite directions. The ball’s velocity increases while angular velocity decreases until \( v = \rho \omega \) when pure rolling starts.
Fig. 19. Five scenarios (1)–(5) of a straight ball motion based on $\rho \omega_0$ relative to $v_0$ shown in the upper left corner.