Rotation in the Space*

(Com S 477/577 Notes)

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The position of a point after some rotation about the origin can simply be obtained by multiplying its coordinates with a matrix. One reason for introducing homogeneous coordinates is to be able to describe translation by a matrix so that multiple transformations, whether each is a rotation or a translation, can be concatenated into one described by the product of their respective matrices. However, in some applications (such as spaceship tracking), we need only be concerned with rotations of an object, or at least independently from other transformations. In such a situation, we often need to extract the rotation axis and angle from a matrix which represents the concatenation of multiple rotations. The homogeneous transformation matrix, however, is not well suited for the purpose.

1 Euler Angles

A rigid body in the space has a coordinate frame attached to itself and located often at the center of mass. This frame is referred to as the *body frame* or *local frame*. The position, orientation, and motion of the body can be described using the body frame relative to a fixed reference frame, called the *world frame*.

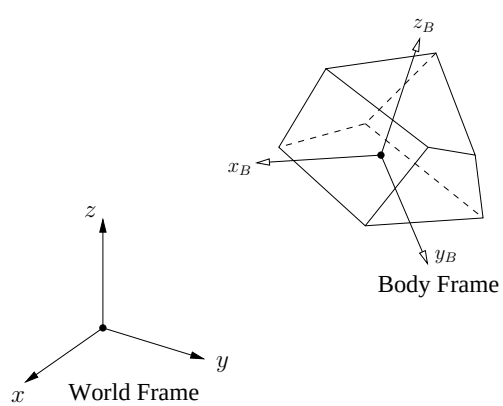


Figure 1: Body frame (local) vs. world frame (global).

The rigid body has six degrees of freedom: its position given by the x, y, z coordinates of its center of mass (i.e., the origin of the local frame) in the world frame, and its orientation described

*The material is partially based on Chapters 3–6 of the book [2]

Figure 2: Yaw, pitch, roll of an aircraft.

by three angles of rotation of its body frame from the world frame. To describe this rotation, one choice is to use three angles respectively about the x-, y-, and z-axes of the world frame. A more convenient choice, from the perspective of the rotating body, is to use three angles about the axes of the body frame. The latter three angles are called *Euler angles*.

For example, an aircraft in flight can perform three independent rotations: *roll*, about an axis from nose to tail (often chosen as the x-axis); *pitch*, nose up or down about an axis from wing to wing (i.e., the y-axis); and *yaw*, nose left or right about an axis from top to bottom (i.e., the z-axis).

There are several conventions for Euler angles, depending on the axes about which the rotations are carried out. Here we introduce the Z-Y-X Euler angles, namely, yaw, pitch, and roll, which are the most often used. The body frame starts in the same orientation as the world frame. To achieve its final orientation, the first rotation is through an angle ϕ, about the body z-axis, the second rotation through an angle $\theta \in [0, \pi]$ about the body y-axis, and the third rotation through an angle ψ about the body x-axis. Here, ϕ, θ, and ψ correspond to as yaw, pitch, and roll, respectively. The three rotations are illustrated in Figure 3, where x'_B-y'_B-z'_B and x''_B-y''_B-z''_B are the two intermediate configurations of the body frame x_B-y_B-z_B.

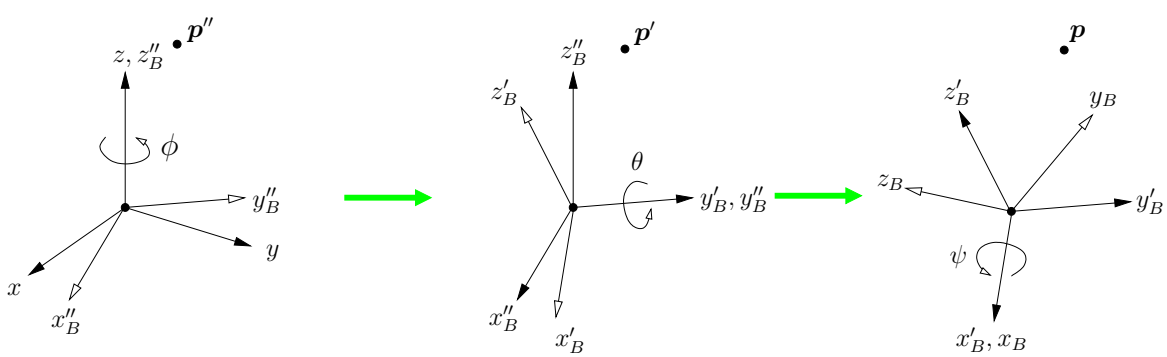


Figure 3: Z-Y-X Euler angles ϕ (yaw), θ (pitch), ψ (roll). Here, $\boldsymbol{p}$, $\boldsymbol{p}'$, and $\boldsymbol{p}''$ are the same point of coordinates in the frames x_B-y_B-z_B, x'_B-y'_B-z'_B, and x''_B-y''_B-z''_B, respectively.

Now we derive a single 3×3 matrix to combines the effects of the three rotations. The matrix, through multiplication, will map a point $\boldsymbol{p}$ with its coordinates expressed in the body frame $x_{\mathcal{B}}$-

$y_{\mathcal{B}}$-$z_{\mathcal{B}}$ to its coordinates in the world frame x-y-z. Let us determine these world coordinates by considering the four frames backward: $x_{\mathcal{B}}$-$y_{\mathcal{B}}$-$z_{\mathcal{B}}$, x'_B-y'_B-z'_B, x''_B-y''_B-z''_B, and x-y-z. Since the frame $x_{\mathcal{B}}$-$y_{\mathcal{B}}$-$z_{\mathcal{B}}$ is obtained from the frame x'_B-y'_B-z'_B via a rotation about the $x'_{\mathcal{B}}$-axis through an angle ψ, $\boldsymbol{p}$ (rigidly attached to $x_{\mathcal{B}}$-$y_{\mathcal{B}}$-$z_{\mathcal{B}}$) has undergone exactly the same rotation. Its coordinates in the frame x'_B-y'_B-z'_B are therefore

$$\boldsymbol{p}' = \mathrm{Rot}_x(\psi)\boldsymbol{p}.$$

Next, the frame x'_B-y'_B-z'_B is obtained from the frame x''_B-y''_B-z''_B after a rotation about the $y''_{\mathcal{B}}$-axis through an angle θ. So the same point has coordinates

$$\boldsymbol{p}'' = \mathrm{Rot}_y(\theta)\boldsymbol{p}' = \mathrm{Rot}_y(\theta)\mathrm{Rot}_x(\psi)\boldsymbol{p}$$

in the frame x''_B-y''_B-z''_B. Similarly, the coordinates of the point in the frame x-y-z is

$$\mathrm{Rot}_z(\phi)\boldsymbol{p}'' = \mathrm{Rot}_z(\phi)\mathrm{Rot}_y(\theta)\mathrm{Rot}_x(\psi)\boldsymbol{p}.$$

Thus, the transformation matrix associated with the Z-Y-X Euler angles are

$$\begin{aligned}
\mathrm{Rot}_{zyx}(\phi,\theta,\psi) &= \mathrm{Rot}_z(\phi)\cdot\mathrm{Rot}_y(\theta)\cdot\mathrm{Rot}_x(\psi) \\
&= \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix} \\
&= \begin{pmatrix} \cos\phi\cos\theta & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi \\ \sin\phi\cos\theta & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi \\ -\sin\theta & \cos\theta\sin\psi & \cos\theta\cos\psi \end{pmatrix} \quad (1)
\end{aligned}$$

Note that the matrix $\mathrm{Rot}_{zyx}(\phi,\theta,\psi)$ maps the body coordinates to the world coordinates. A point with the coordinates $\boldsymbol{q}$ in the world frame x-y-z has the coordinates $\mathrm{Rot}_{zyx}(\phi,\theta,\psi)^T\boldsymbol{q}$ in the body frame x_B-y_B-z_B.

Euler angles are defined in terms of three moving axes. There is a total of $3\cdot 2\cdot 2 = 12$ choices for the triple of axes. To see this, we fix the first axis in the triple to be, say, X, the third axis is either X or not X. Each choice of the third axis further has two possible choices for the second axis, which yields four possible triples: X-Y-X, X-Z-X, X-Z-Y, and X-Y-Z. Essentially, there are only two types of triples: those in which the three axes are different and those in which the first and third coincide.[1]

Singularities happen when two of three axes coincide. Consider the Z-X-Z Euler angles α, β, γ with $\beta = 0$. Here, the first and the third rotations are about the same axis. All the Euler angles $(\alpha, 0, \gamma)$ with the same $\alpha+\gamma$ value thus describe one rotation. Namely, one degree of freedom is lost. Linear interpolation from one orientation to another is not well-behaved. Imagine, when the latitude and longitude values are interpolated. What will happen when latitude goes towards 90 degrees to reach the north pole? All longitude values there make no difference as they end up describing the same point! This is a phenomenon referred to as "gimbal lock", drawing its name from certain orientation with three nested moving gimbals in which two of the three axes become collinear — restricting the available rotations to only two axes. (Watch a nice video tutorial on gimbal lock at `https://www.youtube.com/watch?v=zc8b2Jo7mno`).

[1] The standard rotation package within the software Mathematica uses the sequence Z-X-Z.

The above singularity issue with Euler angles is because they form a 3D box $[0, 2\pi]^3$, while the rotations constitute a 3D projective space. The mapping between the two spaces cannot be "continuous both ways" (or, strictly speaking, a homeomorphism[2]).

2 Arbitrary Rotation Axis

Let $\boldsymbol{v}$ be a vector that is undergoing a rotation of the amount θ about some axis through the origin with an arbitrary orientation. This can be viewed as a rotation about a line that was treated in Appendix A in the notes titled "Transformations in Homogeneous Coordinates". The resulting vector can be found by first rotating the axis to the z-axis, performing the rotation, and rotating the axis back to its original orientation. Now let us present a simpler approach with direct geometric meaning.

Let $\hat{\boldsymbol{\omega}} = (\omega_x, \omega_y, \omega_z)^\top$ be the unit vector along the rotation axis. To obtain the resulting $\boldsymbol{v}'$ from the rotation, we decompose $\boldsymbol{v}$ into two parts: one along $\hat{\boldsymbol{v}}$ and the other in the plane Π containing the origin and perpendicular to $\hat{\boldsymbol{v}}$:

$$\begin{aligned} \boldsymbol{v}_\parallel &= (\boldsymbol{v} \cdot \hat{\boldsymbol{\omega}})\hat{\boldsymbol{\omega}}, \\ &= \boldsymbol{v} + \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{v}) \qquad (2) \\ \boldsymbol{v}_\perp &= \boldsymbol{v} - \boldsymbol{v}_\parallel \\ &= -\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{v}). \qquad (3) \end{aligned}$$

As shown in Figure 4, the component $\boldsymbol{v}_\parallel$ is not affected by rotation. We need only determine the vector $\boldsymbol{v}'_\perp$ that results from applying the same rotation to $\boldsymbol{v}_\perp$.

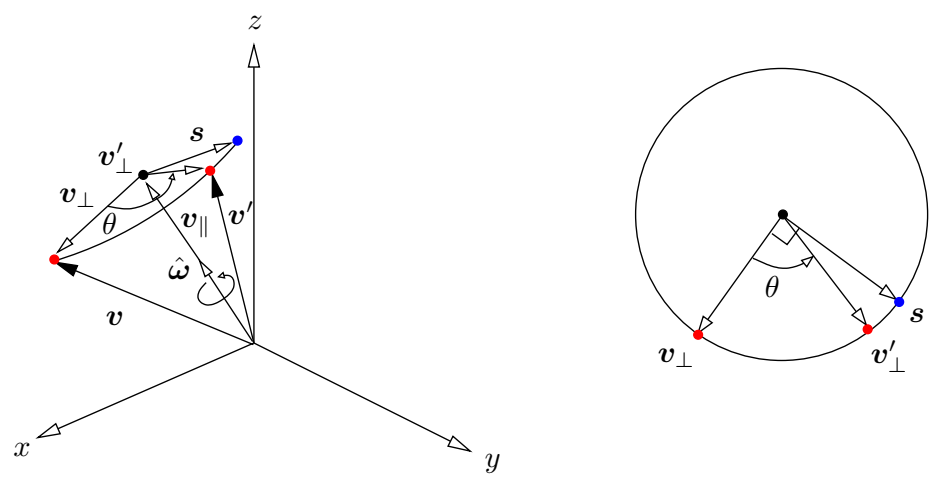


Figure 4: The vector $\boldsymbol{v}$ is rotated to $\boldsymbol{v}'$ about the axis $\hat{\boldsymbol{v}}$.

The plane Π is spanned by two orthogonal vectors: $\boldsymbol{v}_\perp$ and

$$\boldsymbol{s} = \hat{\boldsymbol{\omega}} \times \boldsymbol{v}_\perp = \hat{\boldsymbol{\omega}} \times \boldsymbol{v}.$$

[2] a bijective and continuous function whose inverse is also continuous

It is easy to see that $\boldsymbol{v}_\perp, \boldsymbol{s}, \hat{\boldsymbol{\omega}}$ are the axes of a right-handed system. Since $\hat{\boldsymbol{\omega}}$ is orthogonal to $\boldsymbol{v}_\perp$, $\boldsymbol{v}_\perp$ and $\boldsymbol{s}$ have the same length. The vector $\boldsymbol{v}'_\perp$ lies in the plane Π, and has the form

$$\begin{aligned} \boldsymbol{v}'_\perp &= \boldsymbol{v}_\perp \cos\theta + \boldsymbol{s}\sin\theta \\ &= -\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{v})\cos\theta + \hat{\boldsymbol{\omega}} \times \boldsymbol{v}\sin\theta \end{aligned} \tag{4}$$

Finally, we have the rotated vector from $\boldsymbol{v}$:

$$\begin{aligned} \boldsymbol{v}' &= \boldsymbol{v}_\parallel + \boldsymbol{v}'_\perp \\ &= \boldsymbol{v} + \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{v}) - \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{v})\cos\theta + \hat{\boldsymbol{\omega}} \times \boldsymbol{v}\sin\theta \qquad \text{(by (2) and (4))} \\ &= \boldsymbol{v} + \sin\theta(\hat{\boldsymbol{\omega}} \times \boldsymbol{v}) + (1-\cos\theta)\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{v}). \end{aligned} \tag{5}$$

Let us introduce an operator $[\cdot]_\times$ which applies to a vector $\boldsymbol{a} \in \mathbb{R}^3$ to yield an antimmetric matrix $[\boldsymbol{a}]_\times$ such that $[\boldsymbol{a}]_\times \boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b}$ for every $\boldsymbol{b} \in \mathbb{R}^3$. More specifically, we have

$$[\hat{\boldsymbol{\omega}}]_\times = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$$

Then equation (5) can be rewritten as

$$\boldsymbol{v}' = \left(I_3 + \sin\theta[\hat{\boldsymbol{\omega}}]_\times + (1-\cos\theta)[\hat{\boldsymbol{\omega}}]_\times^2\right)\boldsymbol{v}, \tag{6}$$

where I_3 is the 3×3 identity matrix. The rotation matrix, refererred to as the *Rodrigues' rotation formula*, is given as

$$\begin{aligned} \mathrm{Rot}_{\hat{\boldsymbol{\omega}}}(\theta) &= I_3 + \sin\theta[\hat{\boldsymbol{\omega}}]_\times + (1-\cos\theta)[\hat{\boldsymbol{\omega}}]_\times^2 \qquad (7) \\ &= \begin{pmatrix} \omega_x^2(1-\cos\theta)+\cos\theta & \omega_x\omega_y(1-\cos\theta)-\omega_z\sin\theta & \omega_x\omega_z(1-\cos\theta)+\omega_y\sin\theta \\ \omega_x\omega_y(1-\cos\theta)+\omega_z\sin\theta & \omega_y^2(1-\cos\theta)+\cos\theta & \omega_y\omega_z(1-\cos\theta)-\omega_x\sin\theta \\ \omega_x\omega_z(1-\cos\theta)-\omega_y\sin\theta & \omega_y\omega_z(1-\cos\theta)+\omega_x\sin\theta & \omega_z^2(1-\cos\theta)+\cos\theta \end{pmatrix}. \end{aligned}$$

3 Rotation Matrix

We have seen the use of a matrix to represent a rotation. Such a matrix is referred to as a *rotation matrix*. In this section we look at the properties of rotation matrix. Below let us first review some concepts from linear algebra.

3.1 Eigenvalues

An $n \times n$ matrix A is *orthogonal* if its columns are unit vectors and orthogonal to each other, namely, if $A^\top A = I_n$, where I_n is the $n \times n$ identity matrix. An orthogonal matrix has determinant $\det(A) = \pm 1$.

A complex number λ is called the *eigenvalue* of A if there exists a vector $\boldsymbol{x} \neq 0$ such that $A\boldsymbol{x} = \lambda\boldsymbol{x}$. It tells whether the vector $\boldsymbol{x}$ is stretched or shrunk or reversed or left unchanged — when it is multiplied by A. The vector $\boldsymbol{x}$ is called an *eigenvector* of A associated with the eigenvalue λ. The set of all eigenvalues is called the *spectrum* of A.

The matrix A has exactly n eigenvalues (multiplicities included). They are the roots of the nth degree polynomial $\det(A-\lambda I_n)$, called the *characteristic polynomial*. Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of this polynomial, then

$$\det(A-\lambda I_n) = (-1)^n(\lambda-\lambda_1)^{\delta_1}\cdots(\lambda-\lambda_k)^{\delta_k}, \qquad \text{for some } \delta_1, \ldots, \delta_k > 0.$$

Here δ_i, $1 \le i \le k$, is the *algebraic multiplicity* of λ_i.

The *geometric multiplicity* of λ_i is $n - \operatorname{rank}(A-\lambda_i I_n)$. It specifies the maximum number of linearly independent eigenvectors associated with λ_i.

Example 1. The diagonal matrix aI_n has the characteristic polynomial $(a-\lambda)^n$. So a is the only eigenvalue while every vector $\boldsymbol{x} \in C^n$, $\boldsymbol{x} \neq 0$ is an eigenvector. Both algebraic and geometric multiplicities of a are n.

Example 2. The nth order matrix

$$A = \begin{pmatrix} a & 1 & 0 & \cdots & 0 \\ 0 & a & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & a \end{pmatrix}$$

also has the characteristic polynomial $(a-\lambda)^n$ and a as its only eigenvalue. The algebraic multiplicity of a is n. However, the rank of $A - aI_n$ is $n-1$; hence the geometric multiplicity of a is $n-(n-1)=1$.

The product of the n eigenvalues equals the determinant of A. The sum of the n eigenvalues equals the sum of the n diagonal entries; this number is called the *trace* of A and denoted $\operatorname{Tr}(A)$.

The matrix A is *invertible* if and only if it does not have a zero eigenvalue. A symmetric matrix has real eigenvalues. A symmetric and positive definite matrix has positive eigenvalues.

3.2 Matrices Describing Rotations

All rotation matrices applying to points in the space are 3×3 .

Theorem 1 *A 3×3 matrix describes a rotation about some axis through the origin if and only if it is orthogonal and has determinant 1.*

Proof ($\Rightarrow$) We first show that any rotation matrix R is orthogonal and $\det(R) = 1$. The inner product of two vectors, say, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, must be preserved under the rotation; namely,

$$(R\boldsymbol{v}_1)^\top(R\boldsymbol{v}_2) = \boldsymbol{v}_1^\top\boldsymbol{v}_2.$$

The above implies that

$$\boldsymbol{v}_1^\top(R^\top R - I_3)\boldsymbol{v}_2 = 0.$$

Because $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are arbitrarily chosen, it must hold that

$$R^\top R - I_3 = 0 \qquad \text{and thus} \qquad R^\top R = I_3.$$

Therefore, R is orthogonal. Further more, from the equation

$$\det(R^\top R) = \det(R)^\top \det(R) = \Big(\det(R)\Big)^2 = 1$$

we infer that $\det(R)$ must be 1 or -1. But a rotation preserves the right-handedness of a coordinate frame, so we cannot have $\det(R) = -1$. For example, the diagonal matrix with diagonal entries 1, 1, and -1 would describe a reflection about the x-y plane instead of a rotation. We have thus shown that $\det(R) = 1$.

($\Leftarrow$) Suppose R is a 3×3 orthogonal matrix with determinant 1. We want to show that it represents a rotation. First, we show that there exists a unit vector $\hat{\boldsymbol{\omega}}$ such that $R\hat{\boldsymbol{\omega}} = \hat{\boldsymbol{\omega}}$. This vector will be along the rotation axis. It suffices to establish that the matrix R has an eigenvalue of 1, which is true if and only if $\det(R - I_3) = 0$. We have

$$\begin{aligned}
\det(R - I_3) &= \det(R^\top)\det(R - I_3) \\
&= \det(R^\top R - R^\top) \\
&= \det(I_3 - R^\top) \qquad\qquad (A \text{ orthogonal}) \\
&= \det((I_3 - R)^\top) \\
&= \det(I_3 - R) \\
&= -\det(R - I_3).
\end{aligned}$$

Thus $\det(R - I_3) = 0$. Next, we show that the plane containing the origin and orthogonal to $\hat{\boldsymbol{\omega}}$ maps to itself under the rotation. Consider an arbitrary vector $\boldsymbol{v}$ in this plane, i.e., $\boldsymbol{v}^\top \hat{\boldsymbol{\omega}} = 0$. The vector $R\boldsymbol{v}$ is orthogonal to $\hat{\boldsymbol{\omega}}$ because

$$\begin{aligned}
(R\boldsymbol{v})^\top \hat{\boldsymbol{\omega}} &= \boldsymbol{v}^\top R^\top R \hat{\boldsymbol{\omega}} \\
&= \boldsymbol{v}^\top I \hat{\boldsymbol{\omega}} \\
&= \boldsymbol{v}^\top \hat{\boldsymbol{\omega}} \\
&= 0.
\end{aligned}$$

Finally, the inner product of two vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ is invariant under the transformation because R is orthogonal. Therefore, the lengths of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ and the angles between them are preserved under R. Hence the transformation represented by R is a rotation. $\square$

A rotation R about an axis ℓ not through the origin cannot be simply represented by a 3×3 orthogonal matrix with determinant 1. Below we reason this by contradiction. First, let $\hat{\boldsymbol{\omega}}$ be the direction of the axis, and $\boldsymbol{b}$ be a point on the axis. Since the origin is not on the axis, $\boldsymbol{b} \neq \boldsymbol{0}$. Every point $\boldsymbol{p} \in \mathbb{R}^3$ is transformed under the rotation into $R(\boldsymbol{p} - \boldsymbol{b}) + \boldsymbol{b}$. Assume that the transformation is equivalent to some rotation about the origin. Then there exists a rotation matrix R' such that

$$R(\boldsymbol{p} - \boldsymbol{b}) + \boldsymbol{b} = R'\boldsymbol{p}. \tag{8}$$

For any $\boldsymbol{q} \neq \boldsymbol{p}$, it also holds

$$R(\boldsymbol{q} - \boldsymbol{b}) + \boldsymbol{b} = R'\boldsymbol{q}. \tag{9}$$

Now, subtract (9) from (8):

$$R(\boldsymbol{p} - \boldsymbol{q}) = R'(\boldsymbol{p} - \boldsymbol{q}).$$

Since $\boldsymbol{p}$ and $\boldsymbol{q}$ are arbitrary points, the vector $\boldsymbol{p} - \boldsymbol{q}$ is arbitrary. For the above equation to always hold, $R = R'$ must be true. Equation (8) now reduces to

$$R\boldsymbol{b} = \boldsymbol{b},$$

which implies that the vector $\boldsymbol{b}$ (from the origin) is aligned with the rotation axis ℓ, in other words, ℓ passes through the origin. Thus, a contradiction.

Two successive rotations about the same axis in $\mathbb{R}^3$ is equivalent to one rotation about this axis through an angle which is the sum of the two rotation angles. Even if they are not about the same axis through the origin, their composition is still a rotation through some angle about some axis. The reason is that the product of their rotation matrices, each orthogonal with determinant 1, is still orthogonal with determinant 1. Thus, by Theorem 1 the product matrix describes a rotation. To state this formally, *the composition of any two rotations is equivalent to a rotation.*

3.3 Recovery of Rotation Axis and Angle

Consider the following orthogonal matrix with determinant 1:

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}. \tag{10}$$

Theorem 1 says that it represents a rotation about some axis $\hat{\boldsymbol{\omega}}$ through the origin. How to find the axis $\hat{\boldsymbol{\omega}}$ and angle θ of this rotation? Here we first consider the case that R is symmetric. The matrix also has the form (7) in terms of $\hat{\boldsymbol{\omega}}$ and θ, where by comparing the three pairs of off-diagonal entries that are symmetric about the diagonal, we find that

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \sin\theta = \hat{\boldsymbol{\omega}} \sin\theta = 0.$$

Thus, $\sin\theta = 0$, which implies $\theta = 0$ or π.

- $\theta = 0$. There is no rotation at all. It is meaningless to specify a rotation axis $\hat{\boldsymbol{\omega}}$.
- $\theta = \pi$. The rotation matrix (7) simplifies to

$$\mathrm{Rot}_{\hat{\boldsymbol{\omega}}}(\pi) = \begin{pmatrix} 2\omega_x^2 - 1 & 2\omega_x\omega_y & 2\omega_x\omega_z \\ 2\omega_x\omega_y & 2\omega_y^2 - 1 & 2\omega_y\omega_z \\ 2\omega_x\omega_z & 2\omega_y\omega_z & 2\omega_z^2 - 1 \end{pmatrix}.$$

  Equating the above matrix with (10) yields the following six equations:

$$\begin{aligned} 2\omega_x^2 &= r_{11} + 1, && (11) \\ 2\omega_x\omega_y &= r_{12}, && (12) \\ 2\omega_x\omega_z &= r_{13}, && (13) \\ 2\omega_y^2 &= r_{22} + 1, && (14) \\ 2\omega_y\omega_z &= r_{23}, && (15) \\ 2\omega_z^2 &= r_{33} + 1, && (16) \end{aligned}$$

  If we first obtain ω_x from (11) and then obtain ω_y and ω_z from (12) and (13), respectively, this gives us one solution of the rotation axis:

$$\hat{\boldsymbol{\omega}} = \frac{1}{\sqrt{2(r_{11}+1)}} \begin{pmatrix} r_{11} + 1 \\ r_{21} \\ r_{31} \end{pmatrix}$$

Note that $r_{12} = r_{21}$ and $r_{13} = r_{31}$ due to the symmetry of R. A second solution with the opposite rotation axis is ignored since it just corresponds to a clockwise rotation through π about $\hat{\boldsymbol{\omega}}$. If we start with solving for ω_y and ω_z from (14) and (16), respectively, and then for the remaining two components, we end up with two more feasible rotation axes:

$$\begin{aligned}
\hat{\boldsymbol{\omega}} &= \frac{1}{\sqrt{2(r_{22}+1)}} \begin{pmatrix} r_{12} \\ r_{22}+1 \\ r_{32} \end{pmatrix} \\
\hat{\boldsymbol{\omega}} &= \frac{1}{\sqrt{2(r_{33}+1)}} \begin{pmatrix} r_{13} \\ r_{23} \\ r_{33}+1 \end{pmatrix}
\end{aligned}$$

Now consider the case that R is not symmetric, that is, $R^T \neq R$. It follows from the proof of Theorem 1 that $\hat{\boldsymbol{\omega}}$ is an eigenvector of R which corresponds to the eigenvalue 1. Namely,

$$(R - I_3)\hat{\boldsymbol{\omega}} = 0,$$

from which we can obtain a solution vector by setting one of the non-zero components in $\hat{\boldsymbol{\omega}}$ to 1. But there is a more elegant way. Note that $\hat{\boldsymbol{\omega}}$ is also an eigenvector of $R^\top$ for

$$\begin{aligned}
R^\top \hat{\boldsymbol{\omega}} &= R^\top R \hat{\boldsymbol{\omega}} \\
&= I_3 \hat{\boldsymbol{\omega}} \\
&= \hat{\boldsymbol{\omega}}.
\end{aligned}$$

We have

$$\begin{aligned}
\mathbf{0} &= (R - R^\top)\hat{\boldsymbol{\omega}} \\
&= \begin{pmatrix} 0 & q_3 & -q_2 \\ -q_3 & 0 & q_1 \\ q_2 & -q_1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \\
&= \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
q_1 &= r_{23} - r_{32}, \\
q_2 &= r_{31} - r_{13}, \\
q_3 &= r_{12} - r_{21}.
\end{aligned}$$

The last equation above implies that the rotation axis must be in the same direction as

$$\boldsymbol{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} r_{23} - r_{32} \\ r_{31} - r_{13} \\ r_{12} - r_{21} \end{pmatrix}. \tag{18}$$

We normalize $\boldsymbol{q}$ to obtain the unit vector $\hat{\boldsymbol{\omega}} = (\omega_x, \omega_y, \omega_z)$ to represent the axis of rotation.

Next, we look at how to recover the rotation angle. If the rotation happens to be about the z-axis through an angle θ, then we have the rotation matrix

$$R = \mathrm{Rot}_z(\theta) \equiv \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

And the rotation angle θ satisfies $\cos\theta = r_{11}$. We can also utilize the trace

$$\mathrm{Tr}(R) = r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta$$

and obtain

$$\cos\theta = \frac{\mathrm{Tr}(R) - 1}{2}, \tag{19}$$

$$\begin{aligned} \sin\theta &= \pm\sqrt{1 - \sin^2\cos\theta} \\ &= \pm\sqrt{\frac{3 - \mathrm{Tr}(R)^2 + 2\mathrm{Tr}(R)}{4}}. \end{aligned} \tag{20}$$

The sign of $\sin\theta$ must agree with that of r_{21}.

In fact, the solution of θ from (19)–(20) generalizes to the situation with an arbitrary rotation axis. To see this, we use the old trick of applying a sequence of rotations, represented by a matrix Q, to transform the axis $\hat{\boldsymbol{\omega}}$ into the z-axis. Then we carry out the rotation about the z-axis through the angle θ. Finally, we apply a sequence of reverse rotations $Q^\top$ to transform the z-axis back to $\hat{\boldsymbol{\omega}}$. The rotation matrix is represented as

$$R = Q^\top \mathrm{Rot}_z(\theta) Q. \tag{21}$$

The trace function is commutative, that is, $\mathrm{Tr}(CD) = \mathrm{Tr}(DC)$ for any two square matrices of the same dimensions.[3] This property allows us to derive the following from (21):

$$\begin{aligned} \mathrm{Tr}(R) &= \mathrm{Tr}\Big(Q^\top \mathrm{Rot}_z(\theta) Q\Big) \\ &= \mathrm{Tr}\Big(\Big(Q^\top \mathrm{Rot}_z(\theta)\Big) Q\Big) \\ &= \mathrm{Tr}\Big(Q\Big(Q^\top \mathrm{Rot}_z(\theta)\Big)\Big) \\ &= \mathrm{Tr}\Big(\Big(QQ^\top\Big)\mathrm{Rot}_z(\theta)\Big) \\ &= \mathrm{Tr}\Big(I_3 \mathrm{Rot}_z(\theta)\Big) \\ &= \mathrm{Tr}\Big(\mathrm{Rot}_z(\theta)\Big) \\ &= 1 + 2\cos\theta. \end{aligned}$$

Hence the rotation angle θ still satisfies equation (19) when the rotation axis is arbitrary. How do we determine the sign in (20)? The rotation about $\hat{\boldsymbol{l}}$ through an angle θ is given by the matrix

[3] The proof is directly based on the definition of trace.

in (7). Comparing an off-diagonal element in the matrix R, say r_{21}, with its corresponding entry in $\mathrm{Rot}_{\hat{l}}(\theta)$ gives us

$$\begin{aligned} \sin\theta &= \frac{r_{21} - \omega_x\omega_y(1-\cos\theta)}{\omega_z} \\ &= \frac{2r_{21} - \omega_x\omega_y(3-\mathrm{Tr}(R))}{2\omega_z}. \end{aligned} \tag{22}$$

The angle $\theta \in [0, 2\pi)$ of rotation is thus completely determined from (19) and (22).

The rotation about the axis $\hat{\boldsymbol{\omega}}$ through θ can also be viewed as one about the axis $-\hat{\boldsymbol{\omega}}$ through $-\theta$, yielding a second solution.

Example 3. Here we describe an application of rotations in tracking. Consider a remote object, such as an aircraft, which is being tracked from a station on the earth. We define a local coordinate system at the station with the xy-plane being the tangent plane to the earth. The x- and y-axes point in the directions of the North and East, respectively, and the z-axis points toward the center of the earth. The coordinate system is drawn in Figure 5.

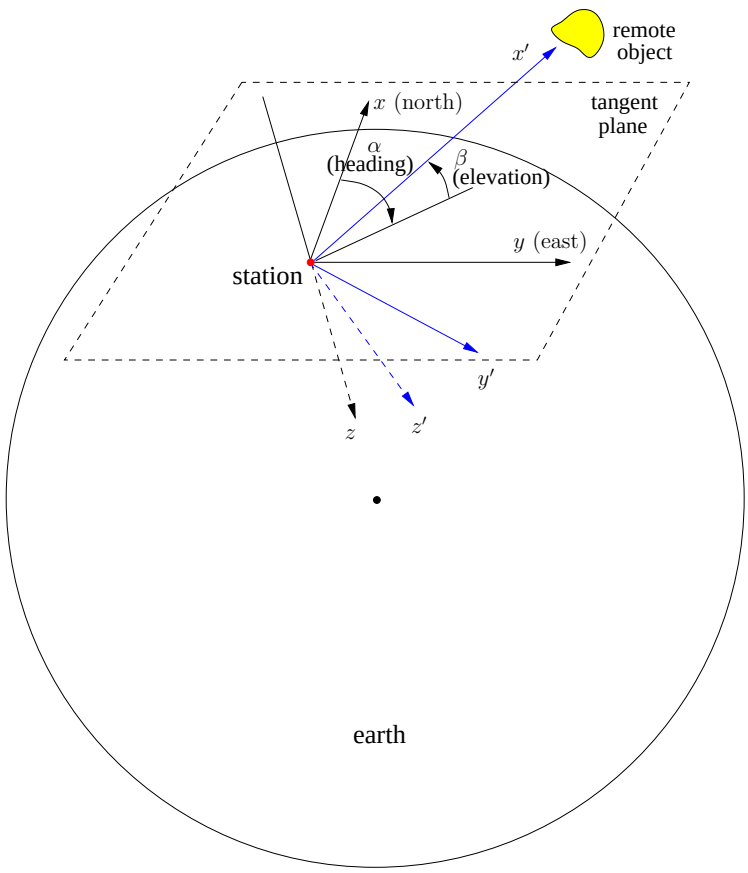


Figure 5: Tracking an aircraft from a ground station.

The *heading* α is the angle in the tangent plane between the North and the projection of the direction to the remote object being tracked. The *elevation* is the angle between the tangent plane and the direction to the object. The tracking transformation is a rotation about the z-axis through the angle α, followed

by a rotation about the new y-axis through the angle β. After these two rotations, the x-axis is pointing toward the object. The tracking coordinate system is made up of the new x'-, y'-, and z'-axes after the two rotations.

Let R be the 3×3 matrix to represent the composite rotation; it relates the tracking coordinate system to the station's coordinate system. Namely, a point $\boldsymbol{p} = (p_x, p_y, p_z)^\top$ in the (re-oriented) tracking frame is $R\boldsymbol{p}$ in the station's frame. Then we have

$$\begin{aligned} R &= \mathrm{Rot}_z(\alpha) \cdot \mathrm{Rot}_y(\beta) \\ &= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \\ &= \begin{pmatrix} \cos\alpha\cos\beta & -\sin\alpha & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta & \cos\alpha & \sin\alpha\sin\beta \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}. \end{aligned}$$

This composite tracking transformation can be represented as an equivalent rotation through some angle θ about some axis in the direction of $\boldsymbol{q}$, which by (18) is given as

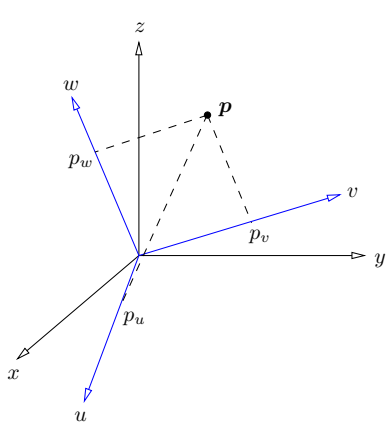
$$\boldsymbol{q} = \begin{pmatrix} \sin\alpha\sin\beta \\ -\cos\alpha\sin\beta - \sin\beta \\ -\sin\alpha\cos\beta - \sin\alpha \end{pmatrix} \equiv \begin{pmatrix} \sin\frac{\alpha}{2}\sin\frac{\beta}{2} \\ -\cos\frac{\alpha}{2}\sin\frac{\beta}{2} \\ -\sin\frac{\alpha}{2}\cos\frac{\beta}{2} \end{pmatrix}. \tag{23}$$

Let $\hat{\boldsymbol{\omega}} = (\omega_x, \omega_y, \omega_z)$ be the normalization of $\boldsymbol{q}$. By (19) and (22) the angle of rotation satisfies the following equations:

$$\begin{aligned} \cos\theta &= \frac{\cos\alpha\cos\beta + \cos\alpha + \cos\beta - 1}{2}, \\ \sin\theta &= \frac{2\sin\alpha\cos\beta - \omega_x\omega_y\big(3 - \mathrm{Tr}(A)\big)}{2\omega_z}. \end{aligned}$$

The domain for the tracking angles is $\alpha \in (-\pi, \pi]$ and $\beta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. As β increases from $\frac{\pi}{2} - \epsilon$ to $\frac{\pi}{2} + \epsilon$, for small ϵ, α instantaneously jumps to $\alpha + \pi$. For this reason, $(\alpha, \frac{\pi}{2})$ is a singular point, and the phenomenon is gimbal lock.

4 Rotation of a Coordinate Frame



We have seen a rotation as an operation carried on a point in the x-y-z frame defined by the following unit vectors:

$$\hat{\boldsymbol{x}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{\boldsymbol{y}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \hat{\boldsymbol{z}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

It can also be used to describe the relative orientation of another frame u-v-w to the x-y-z frame. Denote by $\hat{\boldsymbol{u}}$, $\hat{\boldsymbol{v}}$, and $\hat{\boldsymbol{w}}$ the unit vectors, expressed in the x-y-z frame, which define the axes of the u-v-w frame. Suppose that the two frames initially coincide with each other such that $\hat{\boldsymbol{u}} = \hat{\boldsymbol{x}}$, $\hat{\boldsymbol{v}} = \hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{w}} = \hat{\boldsymbol{z}}$. Then the u-v-w frame rotates about the

origin. A point $\boldsymbol{p} = (p_u, p_v, p_w)^\top$ rigidly attached to this frame rotates as well. Even though its coordinates do not change in the u-v-w frame, its coordinates in the x-y-z frame have become

$$\begin{aligned} \boldsymbol{p}' &= \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \\ &= p_u \hat{\boldsymbol{u}} + p_v \hat{\boldsymbol{v}} + p_w \hat{\boldsymbol{w}} \\ &= (\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}) \begin{pmatrix} p_u \\ p_v \\ p_w \end{pmatrix}. \end{aligned}$$

The matrix

$$R = (\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}})$$

describes the rotation of the u-v-w frame from the x-y-w frame. It maps the coordinates of a point in the first frame to its coordinates in the second frame. This is very useful when considering the u-v-w as attached to a body that is rotating.

Take an arbitrary point, say $\boldsymbol{q} = (q_x, q_y, q_z)^\top$, in the x-y-z frame. What are its coordinates $\boldsymbol{q}' = (q_u, q_v, q_w)^\top$ when expressed in the u-v-w frame? From $\boldsymbol{q} = R\boldsymbol{q}'$ we immediately have that

$$\boldsymbol{q}' = R^{-1}\boldsymbol{q} = R^\top \boldsymbol{q}.$$

5 Rigid Body Transformation in Three Dimensions

A point under a rotation about some axis through a point $\boldsymbol{q}$ is transformed to $R(\boldsymbol{p}-\boldsymbol{q})+\boldsymbol{q}$, where R is the rotation matrix. Rewriting the image point as $R\boldsymbol{p}+(I_3-R)\boldsymbol{q}$, we see that the same rotation can be represented as one about an axis through the origin in the same orientation and through the same angle, followed by a translation. Meanwhile, a transformation in which a translation $\boldsymbol{b}$ precedes a rotation R is equivalent to one in which the same rotation precedes a new translation $R\boldsymbol{b}$, because

$$R(\boldsymbol{p}+\boldsymbol{b}) = R\boldsymbol{p} + R\boldsymbol{b}.$$

Any sequence of translations and rotations can be reordered into a sequence of rotations followed by a sequence of translations. The rotation sequence is equivalent to a single rotation, while the translation sequence is equivalent to a single translation.

A rigid body transformation consists of translations and rotations only. Based on the above reasoning, it is equivalent to a rotation followed by a translation. Namely, it is described as a mapping $T : \boldsymbol{p} \mapsto R\boldsymbol{p} + \boldsymbol{b}$, where $R = (r_{ij})$ is a rotation matrix, and $\boldsymbol{b}$ is a translation vector.

A rigid body transformation in three dimensions is often referred to as a *spatial displacement*. Unless the transformation is a pure rotation, there does not exist a fixed point under such transformation.

Theorem 2 (Chasles' Theorem) *Every spatial displacement $T(\boldsymbol{p}) = R\boldsymbol{p} + \boldsymbol{b}$ is the composition of a rotation about some axis, and a translation along the same axis.*

Instead of proving the theorem, below we describe how to determine the rotation axis ℓ and the translation [3, pp. 19–20]. The axis ℓ is a line in the space called the *screw axis*. Any point on the line is constrained to move along the line under the transformation T.

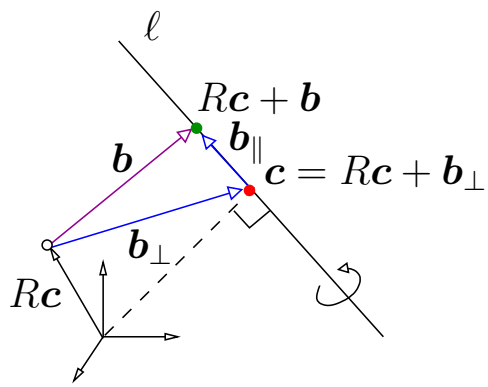


Figure 6: Any rigid body transformation is equivalent to a rotation about some axis, followed by a translation along the same axis.

Following the derivation from Section 3, the direction $\boldsymbol{l}$ of the axis is given in (18). Essentially, $\boldsymbol{l}\times\boldsymbol{u}=(R-R^{\top})\boldsymbol{u}$ for all $\boldsymbol{u}\in\mathbb{R}^3$. We decompose the translation vector $\boldsymbol{b}$ into a part $\boldsymbol{b}_{\parallel}$ along the axis ℓ and a part $\boldsymbol{b}_{\perp}$ orthogonal to it:

$$\boldsymbol{b}=\boldsymbol{b}_{\parallel}+\boldsymbol{b}_{\perp},$$

where

$$\begin{aligned}\boldsymbol{b}_{\parallel} &= \left(\boldsymbol{b}\cdot\frac{\boldsymbol{l}}{\|\boldsymbol{l}\|}\right)\frac{\boldsymbol{l}}{\|\boldsymbol{l}\|}\\ &= \frac{1}{\|\boldsymbol{l}\|^2}(\boldsymbol{b}\cdot\boldsymbol{l})\boldsymbol{l},\\ \boldsymbol{b}_{\perp} &= \boldsymbol{b}-\frac{1}{\|\boldsymbol{l}\|^2}(\boldsymbol{b}\cdot\boldsymbol{l})\boldsymbol{l}.\end{aligned}$$

As shown in Figure 6, $\boldsymbol{c}$ is the perpendicular from the origin to ℓ. It is transformed to the point

$$R\boldsymbol{c}+\boldsymbol{b} \;=\; R\boldsymbol{c}+\boldsymbol{b}_{\perp}+\boldsymbol{b}_{\parallel}.$$

Since $\boldsymbol{c}\perp\ell$, its rotation $R\boldsymbol{c}$ about an axis parallel to ℓ and through the origin must be perpendicular to ℓ as well. The point $R\boldsymbol{c}+\boldsymbol{b}_{\perp}$ must be in the plane containing $\boldsymbol{c}$ and the perpendicular to ℓ. Meanwhile, the same point must also lie on the axis ℓ in order for $R\boldsymbol{c}+\boldsymbol{b}$, its sum with $\boldsymbol{b}_{\parallel}$, to lie on ℓ. We thus infer

$$\boldsymbol{c}=R\boldsymbol{c}+\boldsymbol{b}_{\perp},$$

from which we obtain

$$\boldsymbol{c}=(I_3-R)^{-1}\boldsymbol{b}_{\perp},$$

where the inverse $(I_3-R)^{-1}$ exists due to the uniqueness of $\boldsymbol{c}$. The moment $\boldsymbol{m}$ of ℓ is

$$\begin{aligned}\boldsymbol{l}\times\boldsymbol{c} &= (R-R^{\top})\boldsymbol{c}\\ &= (R-R^{\top})(I_3-R)^{-1}\boldsymbol{b}_{\perp}.\end{aligned}\tag{24}$$

Since

$$\begin{aligned}(R^{\top}+I_3)(I_3-R) &= R^{\top}+I_3-I_3-R\\ &= R^{\top}-R,\end{aligned}$$

we have

$$-(R^\top + I_3) = (R - R^\top)(I_3 - R)^{-1}.$$

Substitute the above into (24):

$$\boldsymbol{m} = \boldsymbol{l} \times \boldsymbol{c} = -(R^\top + I_3)\boldsymbol{b}_\perp. \tag{25}$$

The screw axis is thus given the Plücker coordinates $(\boldsymbol{l}, \boldsymbol{m})$.

References

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[2] J. B. Kuipers. *Quaternions and Rotation Sequences.* Princeton University Press, 1999.

[3] J. M. McCarthy. *An Introduction to Theoretical Kinematics*. The MIT Press, 1990.

[4] `http://sundaram.wordpress.com/2013/03/08/mathematical-reason-behind-gimbal-lock-in-euler-angles/`.