We have learned that the two principal curvatures (and vectors) determine the local shape of a point on a surface. One characterizes the rate of maximum bending of the surface and the tangent direction in which it occurs, while the other characterizes the rate and tangent direction of minimum bending. The rate of surface bending along any tangent direction at the same point is determined by the two principal curvatures according to Euler’s formula.

In this lecture, we will first look at how the local shape at a surface point can be approximated using its principal curvatures and direction. Then we will look at how to characterize the rate of change of a vector defined on a surface with respect to a tangent vector. Our main focus will nevertheless be on two new measures of the curving a surface — its Gaussian and mean curvatures — that turn out to have greater geometrical significance than the principal curvatures.

1 Geometric Interpretation of Principal Curvatures

The values of the principal curvatures and vectors at a point \( p \) on a surface patch \( \sigma \) tell us about the shape near \( p \). To see this, we apply a rigid motion followed by a reparametrization. More specifically, we move the origin to \( p \) and let the tangent plane to \( \sigma \) at \( p \) be the \( xy \)-plane with the \( x \)-axis and \( y \)-axis along the directions of the two principal vectors, which correspond to principal curvatures \( \kappa_1 \) and \( \kappa_2 \), respectively. Furthermore, we let the values of both parameters at the origin be zero, that is,

\[
\sigma(0,0) = 0. \tag{1}
\]

Without any ambiguity, we still denote the new parametrization by \( \sigma \).

Let us determine the function \( z = z(x,y) \) that describes the local shape. The unit principal vectors can be expressed in terms of the partial derivatives:

\[
(1,0,0) = \xi_1 \sigma_u + \eta_1 \sigma_v, \\
(0,1,0) = \xi_2 \sigma_u + \eta_2 \sigma_v.
\]

So can any point \((x,y,0)\) in the tangent plane:

\[
(x,y,0) = x(1,0,0) + y(0,1,0)
\]

---


1The shape does not change under any rigid motion or reparametrization.
\[ \begin{align*}
&= x(\xi_1 \sigma_u + \eta_1 \sigma_v) + y(\xi_2 \sigma_u + \eta_2 \sigma_v) \\
&= s \sigma_u + t \sigma_v, \quad (2) 
\end{align*} \]

where

\[ s = x \xi_1 + y \xi_2 \quad \text{and} \quad t = x \eta_1 + y \eta_2. \quad (3) \]

Let us evaluate \( \sigma(s, t) \) at the parameter values \( s \) and \( t \), applying Taylor’s theorem with higher order terms in \( s \) and \( t \) neglected:

\[ \sigma(s, t) = \sigma(0, 0) + s \sigma_u + t \sigma_v + \frac{1}{2}(s^2 \sigma_{uu} + 2st \sigma_{uv} + t^2 \sigma_{vv}) \]

(by (1) and (2))

All derivatives are evaluated at the origin \( p \). Neglecting the second order terms added to \( x \) and \( y \), the coordinates of \( \sigma(s, t) \) is \((x, y, z)\), where

\[ z = \sigma(s, t) \cdot \hat{n} \]

\[ = \frac{1}{2}(Ls^2 + 2Mst + Nt^2) \]

\[ = \frac{1}{2}(s \ t) \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix}. \]

Writing

\[ T_1 = \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix}, \]

we have from (3):

\[ \begin{pmatrix} s \\ t \end{pmatrix} = xT_1 + yT_2. \]

Thus,

\[ z = \frac{1}{2}(xT_1 + yT_2)^t F_2 (xT_1 + yT_2) \]

\[ = \frac{1}{2} \left( x^2 T_1^t F_2 T_1 + xy(T_1^t F_2 T_2 + T_2^t F_2 T_1) + y^2 T_2^t F_2 T_2 \right) \]

\[ = \frac{1}{2}(\kappa_1 x^2 + \kappa_2 y^2), \]

since \( T_i^t F_2 T_j = \kappa_i \) if \( i = j \) or 0 otherwise. Hence the shape of a surface near the point \( p \) has a quadratic approximation determined by its principal curvature \( \kappa_1 \) and \( \kappa_2 \). It is described by the equation \( z = \frac{1}{2}(\kappa_1 x^2 + \kappa_2 y^2) \).

2 Covariant Derivative

We slightly abuse the notation \( \hat{n} \) to represent a function that assigns to every point \( p \) on the surface \( S \) the normal \( \hat{n}(p) \) at the point. Since \( \hat{n} \) is continuous, it is a vector field on \( S \), and referred to as
the normal vector field. Similarly, \( \hat{t}_1 \) and \( \hat{t}_2 \) are also vector fields on \( S \) that continuously assign to every point two orthogonal principal vectors.

At the point \( p \), a vector field \( Z \) typically changes differently in different tangential directions. The rate of change along a tangent \( w \) is characterized by its covariant derivative along \( w \). More specifically, we let \( \alpha(t) \) be a curve on \( S \) that has initial velocity \( \alpha'(0) = w \). Consider restriction of \( Z \) to \( \alpha \). Then, the covariant derivative of \( Z \) with respect to \( w \) is defined to be

\[
\nabla_w Z = \left. \frac{dZ(\alpha(t))}{dt} \right|_{t=0}.
\]

In particular, consider the \( u \)-curve \( \alpha(u) = \sigma(u, v_0) \) passing through \( p = \sigma(u_0, v_0) \) at velocity \( w = \sigma_u(u_0, v_0) \). We have

\[
\nabla_w Z = \left. \frac{dZ(\alpha(u))}{du} \right|_{u=u_0} = \left. \frac{dZ(\sigma(u, v_0))}{du} \right|_{u=u_0} = Z_u(u_0, v_0).
\]

Reparametrize \( \alpha(u) \) as a unit-speed curve \( \beta(s) \), where \( s \) is arc length. Clearly,

\[
\frac{ds}{du}(0) = \|\alpha'(u_0)\| = \|\sigma_u(u_0, v_0)\|.
\]

At \( p \), let \( \hat{x} = \beta'(0) = \sigma_u(u_0, v_0)/\|\sigma_u(u_0, v_0)\| \) be the unit velocity of the \( u \)-curve. The covariant derivative with respect to \( \hat{x} \) is

\[
\nabla_{\hat{x}} Z = \left. \frac{dZ(\beta(s))}{ds} \right|_{s=0} = \left. \frac{dZ(\alpha(u(s)))}{du} \right|_{u=u_0} = \frac{Z_u(u_0, v_0)}{\|\sigma_u(u_0, v_0)\|}.
\]

In the Darboux frame \( T-V-U \) at \( p \) of a unit-speed surface curve, where \( T \) is the curve tangent, \( U \) the unit surface normal \( \hat{n} \), and \( V = U \times T \), it holds that \( U' = -\kappa_n T - \tau_g V \), where \( \kappa_n \) and \( \tau_g \) are the surface’s normal curvature and curve’s geodesic torsion at \( p \). Meanwhile, \( U' \) is the covariant derivative along \( T \), i.e., \( U' = \nabla_T U \). The normal curvature at the point in the direction \( T \) is equivalently defined to be \( \kappa_n(T) \) [1, p. 196], for we have

\[
\kappa_n(T) = -U' \cdot T = -\nabla_T U \cdot T.
\]

The principal curvatures are the normal curvatures in the two principal directions, that is, the covariant derivatives of the normal with respect to the principal vectors:

\[
\kappa_1 = \kappa_n(\hat{t}_1) = -\nabla_{\hat{t}_1} U \cdot \hat{1} = -\nabla_{\hat{t}_1} \hat{n} \cdot \hat{1},
\]

\[
\kappa_2 = \kappa_n(\hat{t}_2) = -\nabla_{\hat{t}_2} U \cdot \hat{2} = -\nabla_{\hat{t}_2} \hat{n} \cdot \hat{2}.
\]

It can be shown that \( -\nabla_{\hat{t}_i} \hat{n} \cdot \hat{t}_j = 0 \) if \( i \neq j \).
3 Gaussian and Mean Curvatures

Let $\kappa_1$ and $\kappa_2$ be the principal curvatures of a surface patch $\sigma(u, v)$. The Gaussian curvature of $\sigma$ is

$$K = \kappa_1 \kappa_2,$$

and its mean curvature is

$$H = \frac{1}{2}(\kappa_1 + \kappa_2).$$

To compute $K$ and $H$, we use the first and second fundamental forms of the surface:

$$E du^2 + 2Fdudv + Gdv^2 \quad \text{and} \quad L du^2 + 2Mdudv + Ndv^2.$$

Again, we adopt the matrix notation:

$$\mathcal{F}_1 = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \quad \text{and} \quad \mathcal{F}_2 = \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

By definition, the principal curvatures are the eigenvalues of $\mathcal{F}_1^{-1}\mathcal{F}_2$. Hence the determinant of this matrix is the product $\kappa_1\kappa_2$, i.e., the Gaussian curvature $K$. So

$$K = \det(\mathcal{F}_1^{-1}\mathcal{F}_2) = \det(\mathcal{F}_1)^{-1} \det(\mathcal{F}_2) = \frac{LN - M^2}{EG - F^2}. \quad (4)$$

The trace of the matrix is the sum of its eigenvalues, thus, twice the mean curvature $H$. After some calculation, we obtain

$$H = \frac{1}{2} \text{trace}(\mathcal{F}_1^{-1}\mathcal{F}_2) = \frac{1}{2} \frac{LG - 2MF + NE}{EG - F^2}.$$ \quad (5)

An equivalent way to obtain $K$ and $H$ uses the fact that the principal curvatures are also the roots of

$$\det(\mathcal{F}_2 - \kappa \mathcal{F}_1) = 0,$$

which expands into a quadratic equation

$$(EG - F^2)\kappa^2 - (LG - 2MF + NE)\kappa + LN - M^2 = 0.$$$$

The product $K$ and the sum $2H$ of the two roots, can be determined directly from the coefficients. The results are the same as in (4) and (5).

Conversely, given the Gaussian and mean curvatures $K$ and $H$, we can easily find the principal curvatures $\kappa_1$ and $\kappa_2$, which are the roots of

$$\kappa^2 - 2H\kappa + K = 0,$$

i.e., $H \pm \sqrt{H^2 - K}$. 

4
Example 1. We have considered the surface of revolution (see Example 1 in the notes titled “Surface Curvatures”)

\[
\sigma(u,v) = (f(u) \cos v, f(u) \sin v, g(u)),
\]

where we assume, without loss of generality, that \(f > 0\) and \(\dot{f}^2 + \dot{g}^2 = 1\) everywhere. Here a dot denotes \(d/du\). The coefficients of the first and second fundamental forms were determined:

\[
E = 1, \quad F = 0, \quad G = f^2, \quad L = \dot{f}\dot{g} - \ddot{g}, \quad M = 0, \quad N = f\dot{g}.
\]

So the Gaussian curvatures is

\[
K = \frac{LN - M^2}{EG - F^2} = \frac{(\dot{f}\dot{g} - \ddot{g})f\dot{g}}{f^2} = \frac{(\dot{f}\dot{g} - \ddot{g})}{f}.
\]

Meanwhile, differentiate \(\dot{f}^2 + \dot{g}^2 = 1\):

\[
\dot{f}\ddot{f} + \dot{g}\ddot{g} = 0.
\]

Thus,

\[
(\dot{f}\dot{g} - \ddot{g})\dot{g} = -\dot{f}^2\ddot{f} - \ddot{g}^2
\]

\[
= -\dot{f}(\dot{f}^2 + \dot{g}^2)
\]

\[
= -\frac{\ddot{f}}{f}.
\]

So the Gaussian curvature gets simplified to

\[
K = -\frac{\ddot{f}}{f}.
\]

Example 2. Here we compute the Gaussian and mean curvatures of a Monge patch \(z = f(x, y)\). Namely, the patch is described by \(\sigma(x, y) = (x, y, f(x, y))\). First, we obtain the first and second derivatives:

\[
\sigma_x = (1, 0, f_x), \quad \sigma_y = (0, 1, f_y), \quad \sigma_{xx} = (0, 0, f_{xx}), \quad \sigma_{xy} = (0, 0, f_{xy}), \quad \sigma_{yy} = (0, 0, f_{yy}).
\]

Immediately, the coefficients of the first fundamental form are determined

\[
E = 1 + f_x^2, \quad F = f_x f_y, \quad G = 1 + f_y^2.
\]

So is the unit normal to the patch:

\[
\hat{n} = \frac{\sigma_x \times \sigma_y}{\|\sigma_x \times \sigma_y\|} = \frac{(-f_x, -f_y, 1)}{\sqrt{1 + f_x^2 + f_y^2}}
\]

With the normal \(\hat{n}\), we obtain the coefficients of the second fundamental form:

\[
L = \sigma_{xx} \cdot \hat{n} = \frac{f_{xx}}{\sqrt{1 + f_x^2 + f_y^2}},
\]

\[
M = \sigma_{xy} \cdot \hat{n} = \frac{f_{xy}}{\sqrt{1 + f_x^2 + f_y^2}},
\]

\[
N = \sigma_{yy} \cdot \hat{n} = \frac{f_{yy}}{\sqrt{1 + f_x^2 + f_y^2}}.
\]

Plug the expressions for \(E, F, G, L, M, N\) into (4) and (5). A few more steps of symbolic manipulation yield:

\[
K = \frac{LN - M^2}{EG - F^2} = \frac{f_{xx} f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2},
\]

\[
H = \frac{1}{2} \frac{LG - 2MF + NE}{EG - F^2} = \frac{f_{xx}(1 + f_x^2) - 2f_{xy} f_x f_y + f_{yy}(1 + f_x^2)}{2(1 + f_x^2 + f_y^2)^{3/2}}.
\]
4 Classification of Surface Points

The Gaussian curvature is independent of the choice of the unit normal \( \hat{n} \). To see why, suppose \( \hat{n} \) is changed to \( -\hat{n} \). Then the signs of the coefficients of \( L, M, N \) change, so do the signs of both principal curvatures \( \kappa_1 \) and \( \kappa_2 \), which are the roots of \( \det(J^2 - \kappa J_1) \). Their product \( K = \kappa_1 \kappa_2 \) is unaffected. The mean curvature \( H = (\kappa_1 + \kappa_2)/2 \), nevertheless, has its sign depending on the choice of \( \hat{n} \).

The sign of \( K \) at a point \( p \) on a surface \( S \) has an important geometric meaning, which is detailed below.

1. \( K > 0 \) The principal curvatures \( \kappa_1 \) and \( \kappa_2 \) have the same sign. The normal curvature \( \kappa \) in any tangent direction \( t \) is equal to \( \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta \), where \( \theta \) is the angle between \( t \) and the principal vector corresponding to \( \kappa_1 \). So \( \kappa \) has the same sign as that of \( \kappa_1 \) and \( \kappa_2 \). The surface is bending away from its tangent plane in all tangent directions at \( p \). The quadratic approximation of the surface near \( p \) is the paraboloid

\[
z = \frac{1}{2}(\kappa_1 x^2 + \kappa_2 y^2).
\]

We call \( p \) an elliptic point of the surface. The left figure below plots an elliptic paraboloid \( z = x^2 + 2y^2 \) with principal curvatures 2 and 4 at the origin.

2. \( K < 0 \) The principal curvatures \( \kappa_1 \) and \( \kappa_2 \) have opposite signs at \( p \). The quadratic approximation of the surface near \( p \) is a hyperboloid. The point is said to be a hyperbolic point of the surface. The right figure above plots a hyperbolic paraboloid \( z = x^2 - 2y^2 \) with principal curvatures 2 and \(-4\) at the origin.

3. \( K = 0 \) There are two cases:

(a) Only one principal curvature, say, \( \kappa_1 \), is zero. In this case, the quadratic approximation is the cylinder \( z = \frac{1}{2} \kappa_2 y^2 \). The point \( p \) is called a parabolic point of the surface.
(b) Both principal curvatures are zero. The quadratic approximation is the plane \( z = 0 \). The point \( p \) is a planar point of the surface. One cannot determine the shape of the surface near \( p \) without examining the third or higher order derivatives. For example, a point in the plane and the origin of a monkey saddle \( z = x^3 - 3xy^2 \) (shown below) are both planar points, but they have quite different shapes.

A torus is the surface swept by a circle of radius \( a \) originally in the \( yz \)-plane and centered on the \( y \)-axis at a distance \( b, b > a \), from the origin, when the circle revolves about the \( z \)-axis. It is easy to derive the following implicit equation for the torus:

\[
\left( \sqrt{x^2 + y^2} - b \right)^2 + z^2 = a^2.
\]

The torus is a good example which has all three types of points. At points on the outer half of the torus, the torus bends away from its tangent plane; hence \( K > 0 \). At each point on the inner half, the torus bends toward its tangent plane in the horizontal direction, but away from it in the orthogonal direction; hence \( K < 0 \). On the two circles, swept respectively by the top and bottom points of the original circle, every point has \( K = 0 \).

A surface \( S \) is flat if its Gaussian curvature is zero everywhere. A plane is flat. Let it be the \( xy \)-plane with the parametrization \((x, y, 0)\). We can easily show that the plane has zero Gaussian curvature. A circular cylinder, treated in Example 3 of the notes “Surface Curvatures”, has one principal curvature equal to zero and the other equal to the inverse of the radius of its cross section. So a circular cylinder is also flat, even though it is so obviously curved.
A surface is \textit{minimal} provided its mean curvature is zero everywhere. Minimal surfaces have Gaussian curvature $K \leq 0$. This is because $H = (\kappa_1 + \kappa_2)/2 = 0$ implies $\kappa_1 = -\kappa_2$. The catenoid $(\cos u \cosh v, \sin u \cosh v, v)$ plotted over $[0, 2\pi] \times [-2, 2]$ below is a minimal surface.

5 The Gauss Map

The standard unit normal $\hat{n}$ to a surface patch $\sigma$ measures the ‘direction’ of its tangent plane. The change rate of $\hat{n}$ in a tangent direction, i.e., the normal curvature, indicates the degree of variation of surface geometry in that direction at the point. To make the notion of change of geometry independent of any tangent direction, we can measure by the ‘rate of change of $\hat{n}$ per unit area’.

Note that $\hat{n}$ is a point of the unit sphere $S^2$ centered at the origin. The Gauss map from a surface patch $\sigma(u, v) : U \to \mathbb{R}^3$ to the unit sphere $S^2$ sends a point $p = \sigma(u, v)$ to the point $\hat{n}(u, v)$ of $S^2$. The Gauss map may be a many-to-one mapping since multiple points on the patch can have the same unit normal.

Let $R \subseteq U$ be a region in the patch’s domain. The amount by which $\hat{n}$ varies over the corresponding region $\sigma(R)$ on the surface is measured by the area of the image region $N(R)$ on the unit sphere. The rate of change of $\hat{n}$ per unit area is the limit of the ratio of the area $A_N(R)$ of $N(R)$ to the area $A_\sigma(R)$ of the surface region $\sigma(R)$, as $R$ shrinks to a point. To be more precise, we consider $R$ to be a closed disk of radius $\delta$ centered at $(u, v) \in U$. This ratio is

$$\lim_{\delta \to 0} \frac{A_N(R)}{A_\sigma(R)}$$

It can be shown [2, pp. 166–168] that the above ratio is the absolute value of the Gaussian curvature
at \( p \), i.e.,

\[
\lim_{\delta \to 0} \frac{A_N(R)}{A\sigma(R)} = |K|.
\]

The integral of the Gaussian curvature \( K \) over a surface \( S \),

\[
\int \int_S KdS,
\]

is called the total Gaussian curvature of \( S \). It is the algebraic area of the image of the region on the unit sphere under the Gauss map. Note the use of the word ‘algebraic’ since Gaussian curvature can be either positive or negative.

Suppose the patch \( S = \sigma(u, v) \) is defined over the domain \([a, b] \times [c, d] \). Then the total Gaussian curvature is computed as

\[
\int_a^b \int_c^d K(u, v) \sqrt{EG - F^2} \, du \, dv.
\]

**Example 3.** If the Gaussian curvature \( K \) of a surface \( S \) is constant, then the total Gaussian curvature is \( K \cdot A(S) \), where \( A(S) \) is the area of the surface. Thus a sphere of radius \( r \) has total Gaussian curvature \( \frac{1}{r^2} \cdot 4\pi r^2 = 4\pi \), which is independent of the radius \( r \).

**Example 4.** Without any computation, we can determine that an ellipsoid also has total curvature \( 4\pi \). The Gauss map is bijective (one-to-one and onto) since every point on the ellipsoid has a distinct normal. The image region covers the unit sphere. Because the Gaussian curvature is everywhere positive on the ellipsoid, the area of the unit sphere, \( 4\pi \), is the total Gaussian curvature of the ellipsoid.

**References**
