Interpolation by Spline Functions

Com S 477/577

Sep 11, 2003

High-degree polynomials tend to have large oscillations which are not the characteristics of the original data. To yield smooth interpolating curves, cubic spline functions are often used. Roughly speaking, a cubic spline is a set of polynomials of degree three that are smoothly connected at given supporting points.

More precisely, let \( a = x_0 < x_1 < \cdots < x_n = b \) be a partition of the interval \([a, b]\). They are also refer to as knots. A cubic spline \( S \) on the interval is a real function with the following properties:

(a) \( S \) is twice continuously differentiable on \([a, b]\), or in short, \( S \in C^2[a, b] \);

(b) \( S \) coincides with a cubic polynomial on every subinterval \([x_i, x_{i+1}]\), \( i = 0, 1, \ldots, n - 1 \).

Thus a cubic spline consists of cubic polynomials pieced together in such a fashion that their values and those of their first two derivatives coincide at the knots \( x_0, x_1, \ldots, x_n \).

1 Theoretical Foundations

Consider a set \( F = \{f_0, f_1, \ldots, f_n\} \) of \( n + 1 \) real numbers. We denote by \( S(F; \cdot) \) an interpolating spline function with \( S(F; x_i) = f_i \) for \( i = 0, 1, \ldots, n \).

Such an interpolating spline function is not uniquely determined by the supporting points \((x_i, f_i), i = 0, 1, \ldots, n\). Roughly speaking, there are still two degrees of freedom left. So we add an additional requirement:

\[
S''(F; a) = S''(F; b) = 0. \tag{1}
\]

This condition will ensure the uniqueness of the interpolating function.

For this purpose, we consider the sets \( \mathcal{K}^m[a, b] \), integer \( m > 0 \), of real functions \( f \) on \([a, b]\) for which \( f^{(m-1)} \) is absolutely continuous\(^2\) on \([a, b]\) and the square of \( f^{(m)} \) is integrable on the interval. Clearly, \( S \in \mathcal{K}^3[a, b] \). If \( f \in \mathcal{K}^2[a, b] \), then we can define

\[
\|f\|^2 = \int_a^b |f''(x)|^2 \, dx.
\]

The following theorem states that spline functions have an important minimum-norm property.

**Theorem 1** Given a partition \( a = x_0 < x_1 < \cdots < x_n = b \) of the interval \([a, b]\), values \( F = \{f_0, \ldots, f_n\} \), and a function \( f \in \mathcal{K}^3[a, b] \) with \( f(x_i) = f_i \) for \( i = 0, 1, \ldots, n \), then

\[
\|f - S(F; \cdot)\|^2 = \|f\|^2 - \|S(F; \cdot)\|^2 \geq 0
\]

\(^1\)Another alternative is to require that \( S^{(k)}(F; a) = S^{(k)}(F; b) \), for \( k = 0, 1, 2 \); that is, \( S(F; \cdot) \) is periodic. A third alternative is to specify the values of \( S(F; a) \) and \( S'(F; b) \).

\(^2\)A real function \( f \) is absolutely continuous on \([a, b]\) if for every \( \varepsilon > 0 \) there exist \( \delta > 0 \) such that \( \sum |f(b_i) - f(a_i)| < \varepsilon \) for every finite set of intervals \([a_i, b_i]\) with \( a \leq a_1 < b_1 < \cdots < a_n < b_n \leq b \) and \( \sum |b_i - a_i| < \delta \).
holds for every spline function \( S(F; \cdot) \), under the condition that \( S''(F; a) = S''(F; b) = 0 \).

The minimum-norm property of the spline function implies that, among all functions \( f \) in \( K^2[a, b] \) with \( f(x_i) = f_i, \ i = 0, 1, \ldots, n \), the spline function \( S(F; \cdot) \) with \( S''(F; a) = S''(F; b) = 0 \) minimizes the integral

\[
\|f\|^2 = \int_a^b |f''(x)|^2 \, dx.
\]

The spline function under condition (1) is often referred to as the natural spline function.

2 Computing Interpolating Cubic Spline Functions

Now we describe how to determine cubic spline functions that assume prescribed values \( f_0, f_1, \ldots, f_n \) at the knots \( a = x_0 < x_1 < \cdots < x_n = b \), respectively, and satisfy condition (1). Let

\[
h_{j+1} = x_{j+1} - x_j, \quad j = 0, 1, \ldots, n - 1.
\]

For convenience, we adopt a short-hand notation

\[
M_j = S''(F; x_j), \quad j = 0, 1, \ldots, n - 1,
\]

where \( F = \{f_0, f_1, \ldots, f_n\} \), for the values of the second derivatives at the knots. We refer to \( M_0, \ldots, M_n \) as the moments of \( S(F; \cdot) \). As we will see below, spline functions are readily characterized by their moments, which can be calculated as the solution of a system of linear equations.

The second derivative \( S''(F; \cdot) \) of the spline function coincides with a linear function in each interval \([x_j, x_{j+1}]\), \( j = 0, \ldots, n - 1 \). Thus these linear functions can be described in terms of the moments \( M_i \):

\[
S''(F; x) = M_j \frac{x_{j+1} - x}{h_{j+1}} + M_{j+1} \frac{x - x_j}{h_{j+1}} \quad \text{for} \quad x \in [x_j, x_{j+1}].
\]

Integrating the above equation yields

\[
S'(F; x) = -M_j \frac{(x_{j+1} - x)^2}{2h_{j+1}} + M_{j+1} \frac{(x - x_j)^2}{2h_{j+1}} + A_j,
\]

\[
S(F; x) = M_j \frac{(x_{j+1} - x)^3}{6h_{j+1}} + M_{j+1} \frac{(x - x_j)^3}{6h_{j+1}} + A_j(x - x_j) + B_j \quad \text{for} \quad x \in [x_j, x_{j+1}], \ j = 0, \ldots, n - 1.
\]

Here \( A_j, B_j \) are constants of integration for which we can obtain the following equations from \( S(F; x_j) = f_j \) and \( S(F; x_{j+1}) = f_{j+1} \):

\[
M_j \frac{h_{j+1}^2}{6} + B_j = f_j,
\]

\[
M_{j+1} \frac{h_{j+1}^2}{6} + A_j h_{j+1} + B_j = f_{j+1}.
\]

Solving the above equations we have

\[
B_j = f_j - M_j \frac{h_{j+1}^2}{6},
\]

\[
A_j = \frac{f_{j+1} - f_j}{h_{j+1}} - \frac{h_{j+1}}{6}(M_{j+1} - M_j).
\]
This yields the following representation of the spline function in terms of its moments:

\[
S(F; x) = \alpha_j + \beta_j(x - x_j) + \gamma_j(x - x_j)^2 + \delta_j(x - x_j)^3
\]

(5)

for \( x \in [x_j, x_{j+1}] \), where

\[
\alpha_j = f_j,
\]

\[
\gamma_j = \frac{M_j}{2},
\]

\[
\beta_j = S'(F; x_j) = -\frac{M_j h_{j+1}}{2} + A_j
\]

\[
= \frac{f_{j+1} - f_j}{h_{j+1}} - \frac{2M_j + M_{j+1}h_{j+1}}{6},
\]

\[
\delta_j = \frac{S''(F, x_j^+)}{6}
\]

\[
= \frac{M_{j+1} - M_j}{6h_{j+1}}.
\]

Thus the interpolating spline function has been characterized by its moments \( M_j \). Next, we will address how to calculate these moments. The continuity of \( S'(F; \cdot) \) ensures that

\[
S'(F; x_j^-) = S'(F; x_j^+).
\]

(6)

This yields \( n-1 \) equations for the moments. Substituting equations (4) for \( A_j \) in (2) gives

\[
S'(F; x) = -M_j \left( \frac{(x_{j+1} - x)^2}{2h_{j+1}} \right) + M_{j+1} \left( \frac{(x - x_j)^2}{2h_{j+1}} \right) + \frac{f_{j+1} - f_j}{h_{j+1}} - \frac{h_{j+1}}{6}(M_{j+1} - M_j).
\]

For \( j = 1, 2, \ldots, n-1 \), we have therefore

\[
S'(F; x_j^-) = \frac{f_j - f_{j-1}}{h_j} + \frac{h_j}{3} M_j + \frac{h_j}{6} M_{j-1},
\]

\[
S'(F; x_j^+) = \frac{f_{j+1} - f_j}{h_{j+1}} - \frac{h_{j+1}}{3} M_j - \frac{h_{j+1}}{6} M_{j+1},
\]

Plugging these two equations into (6) yields

\[
\frac{h_j}{6} M_{j-1} + \frac{h_j + h_{j+1}}{3} M_j + \frac{h_j + h_{j+1}}{6} M_{j+1} = \frac{f_{j+1} - f_j}{h_{j+1}} - \frac{f_j - f_{j-1}}{h_j}
\]

for \( i = 1, 2, \ldots, n-1 \). The above equations now can be written in a common format:

\[
\mu_j M_{j-1} + 2M_j + \lambda_j M_{j+1} = d_j, \quad j = 1, 2, \ldots, n-1,
\]

where the coefficients are as follows,

\[
\lambda_j = \frac{h_{j+1}}{h_j + h_{j+1}},
\]

\[
\mu_j = 1 - \lambda_j
\]

\[
= \frac{h_j}{h_j + h_{j+1}},
\]

\[
d_j = \frac{6}{h_j + h_{j+1}} \left( \frac{f_{j+1} - f_j}{h_{j+1}} - \frac{f_j - f_{j-1}}{h_j} \right),
\]

3
There are \( n - 1 \) equations for the \( n + 1 \) unknown moments. Two further equations can be derived from the moments at \( a \) and \( b \):

\[
M_0 = S''(F; a) = 0; \quad M_n = S''(F; b) = 0.
\]

So, we let

\[
\lambda_0 = 0, \quad d_0 = 0, \quad \mu_n = 0, \quad d_n = 0.
\]

This leads to the following system of linear equations for the moments:

\[
\begin{pmatrix}
2 & \lambda_0 & 0 \\
\mu_1 & 2 & \lambda_1 \\
& \ddots & \ddots \\
0 & \cdots & 2 & \lambda_{n-1} \\
& \cdots & \cdots & \mu_n \\
\end{pmatrix}
\begin{pmatrix}
M_0 \\
M_1 \\
\vdots \\
M_{n-1} \\
M_n
\end{pmatrix}
= \begin{pmatrix}
d_0 \\
d_1 \\
\vdots \\
d_{n-1} \\
d_n
\end{pmatrix}.
\]

The coefficients \( \lambda_i, \mu_i, d_i \) are well defined. Note in particular that

\[
\lambda_i \geq 0, \quad \mu_i \geq 0, \quad \lambda_i + \mu_i = 1
\]

for all \( i = 0, 1, \ldots, n \) and that these coefficients depend only on the knots \( x_j \) and not on the prescribed values \( f_i \).

**Theorem 2** The system (7) of linear equations is nonsingular for any partition \( a = x_0 < x_1 < \cdots < x_n = b \) of \( [a, b] \).

The above theorem implies that the system (7) of equations has unique solutions. Consequently, the problem of interpolation by cubic splines has a unique solution under the conditions \( S''(F; a) = S''(F; b) = 0 \).

**Example.** Figure 1 shows the interpolation of two functions. In (a), the function \( f(x) = 4 \sin x \) is interpolated with the partition \( x_i = \frac{i \cdot 2\pi}{11}, i = 0, 1, \ldots, 10 \), by a cubic spline consisting of the following ten polynomials:

\[
\begin{align*}
80.3062 & + 27.792x & + & 3.25779x & + & 0.138265x^3, \\
-125.474 & - 70.407x & - & 12.3796x^2 & - & 0.691325x^3, \\
5.7799 & + 13.0979x & + & 5.35209x^2 & + & 0.562363x^3, \\
3.94258 & + 11.3434x & + & 4.79361x^2 & + & 0.503679x^3, \\
4.44089 \cdot 10^{-16} & + & 3.81363x & - & 0.513556x^3, \\
& & & & & - 3.81363x & - 1.18993 \cdot 10^{-16}x^2 & - 0.513556x^3, \\
-3.94258 & + 11.3434x & - & 4.79361x^2 & + & 0.503679x^3, \\
-5.7799 & + 13.0979x & - & 5.35209x^2 & + & 0.562363x^3, \\
125.474 & - 70.407x & + & 12.3796x^2 & - & 0.691325x^3, \\
-80.3062 & + 27.792x & - & 3.25779x & + & 0.138265x^3.
\end{align*}
\]

In (b), the quartic polynomial \( p(x) = \frac{x^4}{4} \) is interpolated with the partition \( -4 < -2 < 0 < 2 < 4 \) by a cubic spline consisting of the following four polynomials:

\[
\begin{align*}
14.6286 & + 21.9429x & + & 9.25714x^2 & + & 0.771429x^3, \\
& & & & & -1.71429x^2 & - 1.05714x^3, \\
& & & & & -1.11022 \cdot 10^{-16}x & - 1.71429x^2 & + 1.05714x^3, \\
\end{align*}
\]
Figure 1: Cubic spline interpolations of (a) \( f(x) = 4 \sin x \) over \([-\frac{5\pi}{2}, \frac{5\pi}{2}]\) and (b) \( p(x) = \frac{x^4}{16} \) over \([-4, 4]\). The interpolating cubic splines are drawn in green lines.

References