Curvature

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We want to find a measure of how ‘curved’ a curve is. Since this “curvature” should depend only on the ‘shape’ of the curve. It should not be changed when the curve is reparametrized. Further, the measure of curvature should agree with our intuition in simple special cases. Straight lines themselves have zero curvature. Large circles should have smaller curvature than small circles which bend more sharply.

The (signed) curvature of a curve parametrized by its arc length is the rate of change of direction of the tangent vector. The absolute value of the curvature is a measure of how sharply the curve bends. Curves which bend slowly, which are almost straight lines, will have small absolute curvature. Curves which swing to the left have positive curvature and curves which swing to the right have negative curvature. The curvature of the direction of a road will affect the maximum speed at which vehicles can travel without skidding, and the curvature in the trajectory of an aeroplane will affect whether the pilot will suffer “blackout” as a result of the g-forces involved.

In this lecture we will primarily look at the curvature of plane curves. The results will be extended to space curves in the next lecture.

1 Tangent and Normal

The standard method of studying the geometry of a curve at a point is to attach orthonormal vectors to the point and see how the directions of these vectors change as the point moves on the curve for an infinitesimal distance. We choose tangent and normal vectors at a regular point.

Let $\alpha(t) = (x(t), y(t))$ be a curve. At a regular point $\alpha(t)$ we have a (non-zero) tangent vector $\alpha'(t) = (x'(t), y'(t))$. So the tangent vector represents the velocity of the curve at the point. The normal vector at $\alpha(t)$ is given by rotating the tangent vector counterclockwise through an angle $\frac{\pi}{2}$. It is given by $(-y'(t), x'(t))$. Note that $(x'(t), y'(t)) \times (-y'(t), x'(t)) = (x'(t))^2 + (y'(t))^2 > 0$.

If $\alpha(t)$ is a unit-speed curve, then both the tangent vector and the normal vector are unit vectors. By convention they are denoted as $T$ and $N$, respectively, with the cross product $T \times N = 1$. 

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For a parametric curve we have a tangent line and a normal line at each regular point \( \alpha(t) \). The tangent line to the curve at \( \alpha(t) \) passes through \( \alpha(t) \) and is parallel to \( \alpha'(t) \neq 0 \). So it has the parametric equation

\[
(x(s), y(s)) = \alpha(t) + s\alpha'(t), \quad s \in (-\infty, \infty),
\]

or equivalently, the algebraic equation

\[
\left((x, y) - \alpha(t)\right) \cdot \left((-y'(t), x'(t))\right) = 0.
\]

The normal line at \( \alpha(t) \) passes through the point and is parallel to \((-y'(t), x'(t))\). So its equations are of the form

\[
(x(s), y(s)) = \alpha(t) + s\left((-y'(t), x'(t))\right), \quad s \in (-\infty, \infty),
\]

or equivalently,

\[
\left((x(s), y(s)) - \alpha(t)\right) \cdot \alpha'(t) = 0.
\]

**Example 1.** Find the tangent and normal lines of the crunodal cubic\(^1\)

\[
\alpha(t) = (t^2 - 1, t(t^2 - 1))
\]

at the points \( t = \pm 1, 0 \).

We obtain that

\[
\begin{align*}
\alpha'(t) &= (2t, 3t^2 - 1), \\
\alpha'(1) &= (2, 2), \\
\alpha'(-1) &= (-2, 2), \\
\alpha'(0) &= (0, -1), \\
\alpha'(\pm 1) &= (0, 0).
\end{align*}
\]

\(^1\)This figure is taken from [2, p. 60].
Here $\alpha = (0,0)$ is referred to as a *double point* since it is attained at both $t = 1$ and $t = -1$. The tangent lines at this double point are respectively

\[(x, y) = s(1,1), \quad \text{or equivalently,} \quad y = x,\]

and

\[(x, y) = s(-1,1), \quad \text{or equivalently,} \quad y = -x.\]

The normal lines at the double point are respectively

\[(x, y) = s(-1,1), \quad \text{or equivalently,} \quad y = -x,\]

and

\[(x, y) = s(-1,-1), \quad \text{or equivalently,} \quad y = x.\]

At $t = 0$, we have $\alpha'(0) = (0,-1)$, and the tangent line at $\alpha(0)$ is

\[(x, y) = (-1,0) + s(0,-1), \quad \text{or equivalently,} \quad x = -1.\]

The normal line at $\alpha(0)$ is

\[(x, y) = (-1,0) + s(1,0), \quad \text{or equivalently,} \quad y = 0.\]

## 2 Curvature

To introduce the definition of curvature, in this section we consider that $\alpha(s)$ is a *unit-speed curve*, where $s$ is the arc length. The *slope angle* $\phi$ is measured counterclockwise from the $x$-axis to the unit tangent $T = \alpha'(s)$, as shown below.

The curvature $\kappa$ of $\alpha$ is the rate of change of direction at that point of the tangent line with respect to arc length, that is,

\[\kappa = \frac{d\phi}{ds}. \tag{1}\]

The *absolute curvature* of the curve at the point is the absolute value $|\kappa|$.

Since $\alpha$ has unit speed, $T \cdot T = 1$. Differentiating this equation yields

\[T' \cdot T = 0.\]

The change of $T(s)$ is orthogonal to the tangential direction, so it must be along the normal direction. The curvature is also defined to measure the turning of $T(s)$ along the direction of the unit normal $N(s)$ where $T(s) \times N(s) = 1$. That is,

\[T' = \frac{dT}{ds} = \kappa N. \tag{2}\]
We can easily derive one of the curvature definitions (1) and (2) from the other. For instance, if we start with (2), then

\[
\kappa = \frac{d}{ds} \cdot N = \lim_{\Delta s \to 0} \frac{T(s + \Delta s) - T(s)}{\Delta s} \cdot N = \lim_{\Delta s \to 0} \frac{\Delta \psi \cdot \|T\|}{\Delta s} = \lim_{\Delta s \to 0} \frac{\Delta \psi}{\Delta s} = \frac{d\phi}{ds}.
\]

**Example 1.** Let us compute the curvature of the unit-speed circle

\[
\alpha(s) = r \left( \cos \frac{s}{r}, \sin \frac{s}{r} \right).
\]

We obtain that

\[
T = \alpha'(s) = \left( -\sin \frac{s}{r}, \cos \frac{s}{r} \right), \quad N = \left( -\cos \frac{s}{r}, -\sin \frac{s}{r} \right), \quad T' = \alpha''(s) = -\frac{1}{r} \left( \cos \frac{s}{r}, \sin \frac{s}{r} \right) = \frac{1}{r} N.
\]

Thus

\[
\kappa(s) = \frac{1}{r}, \quad \text{cf. (2)}
\]

The curvature of a circle equals the inverse of its radius everywhere.

The next result shows that a unit-speed plane curve is essentially determined once we know its curvature at each point of the curve. The meaning of ‘essentially’ here is ‘up to a rigid motion\(^2\) of \(\mathbb{R}^2\).

**Theorem 1** Let \(\kappa : (a, b) \to \mathbb{R}\) be an integrable function. Then there exists a unit-speed curve \(\alpha : (a, b) \to \mathbb{R}^2\) whose curvature is \(\kappa\).

\(^2\)A rigid motion consists of a rotation and a translation.
Proof  Fix $s_0 \in (a, b)$ and define, for any $s \in (a, b)$,
\[
\phi(s) = \int_{s_0}^{s} \kappa(u) \, du, \quad \text{cf. (1)},
\]
\[
\alpha(s) = \left( \int_{s_0}^{s} \cos \phi(t) \, dt, \int_{s_0}^{s} \sin \phi(t) \, dt \right).
\]
Then, the tangent vector of $\alpha$ is
\[
\alpha'(s) = \left( \cos \phi(s), \sin \phi(s) \right),
\]
which is a unit vector making an angle $\phi(s)$ with the $x$-axis. Thus $\alpha$ is unit speed, and has curvature
\[
\frac{d\phi}{ds} = \frac{d}{ds} \int_{s_0}^{s} \kappa(u) \, du = \kappa(s).
\]

The above theorem shows that we can find a plane curve with any given smooth function as its signed curvature. But simple curvature can lead to complicated curves, as shown in the next example.

Example 2.  Let the signed curvature be $\kappa(s) = s$. Following the proof of Theorem 1, and taking $s_0 = 0$, we get
\[
\phi(s) = \int_{0}^{s} u \, du = \frac{s^2}{2},
\]
\[
\alpha(s) = \left( \int_{0}^{s} \cos \frac{s^2}{2} \, ds, \int_{0}^{s} \sin \frac{s^2}{2} \, ds \right).
\]
These integrals can only be evaluated numerically.\(^3\) The curve is drawn in the figure below.\(^4\)

\(^3\)They arise in the theory of diffraction of light, where they are called Fresnel’s integrals, and the curve is called Cornu’s Spiral, although it was first considered by Euler.

\(^4\)Taken from [3, p. 33].
When the curvature $\kappa(s) > 0$, the center of curvature lies along the direction of $N(s)$ at distance $\frac{1}{\kappa}$ from the point $\alpha(s)$. When $\kappa(s) < 0$, the center of curvature lies along the direction of $-N(s)$ at distance $-\frac{1}{\kappa}$ from $\alpha(s)$. In either case, the center of curvature is located at

$$\alpha(s) + \frac{1}{\kappa(s)} N(s).$$

The osculating circle, when $\kappa \neq 0$, is the circle at the center of curvature with radius $\frac{1}{|\kappa|}$, which is called the radius of curvature. The osculating circle approximates the curve locally up to the second order.

The total curvature over a closed interval $[a, b]$ measures the rotation of the unit tangent $T(s)$ as $s$ changes from $a$ to $b$:

$$\Phi(a, b) = \int_a^b \kappa \, ds = \int_a^b \frac{d\phi}{ds} \, ds = \int_a^b d\phi = \phi(b) - \phi(a).$$

If the total curvature over $[a, b]$ is within $[0, 2\pi]$, it has a closed form:

$$\Phi(a, b) = \begin{cases} \arccos \left( T(a) \cdot T(b) \right), & \text{if } T(a) \times T(b) \geq 0; \\ 2\pi - \arccos \left( T(a) \cdot T(b) \right), & \text{otherwise.} \end{cases}$$

When the tangent makes several full revolutions\textsuperscript{5} as $s$ increases from $a$ to $b$, it cannot be determined just from $T(a)$ and $T(b)$.

\textsuperscript{5}For example, the curve is the Cornu’s Spiral.
A point $s$ on the curve $\alpha$ is *simple inflection*, or *inflection*, if the curvature $\kappa(s) = 0$ but $\kappa'(s) \neq 0$. Intuitively, a simple inflection is where the curve swing from the left of the tangent at the point to its right; or in the case of simple closed curve, it is where the closed curve $\alpha$ changes from convex to concave or from concave to convex. In the figure below, the curve on the left has one simple inflection while the curve on the right has six simple inflections.

In general, a point $s$ with $\kappa(s) = \kappa'(s) = \cdots = \kappa^{(j-1)}(s) = 0$ and $\kappa^{(j)}(s) \neq 0$ is an *inflection point of order* $j$. A second order inflection point, also referred to as a point of *simple undulation*, will not alter the convexity or concavity of its neighborhood on a simple closed curve.

A *simple vertex*, or a *vertex*, of a curve satisfies $\kappa' = 0$ but $\kappa'' \neq 0$. Intuitively, a simplex vertex is where the curvature attains a local minimum or maximum. For example, an ellipse has four vertices, on its major and minor axes.

### 3 Curvature of Arbitrary-Speed Curves

Let $\alpha(t)$ be a regular curve but not necessarily unit-speed. We obtain the unit tangent as $T = \alpha' / \|\alpha'\|$ and the unit normal $N$ as the counterclockwise rotation of $T$ by $\frac{\pi}{2}$. Still denote by $\kappa(t)$ the curvature function. Let $\tilde{\alpha}(s)$ be the unit-speed reparametrization of $\alpha$, where $s$ is an arc-length function for $\alpha$. Let $\tilde{T} = d\tilde{\alpha}/ds$ be the unit tangent and $\tilde{\kappa}(s)$ the curvature function under this unit-speed parametrization. The curvature at a point is *independent* of any parametrization so $\kappa(t) = \tilde{\kappa}(s(t))$. Also by definition $T(t) = \tilde{T}(s)$. Differentiate this equation and apply the chain
rule:

\[ T'(t) = \mathbf{T}'(s) \cdot \frac{ds}{dt}. \]  

(3)

Since \( \mathbf{\alpha}(s) \) is unit-speed, we know that

\[ \mathbf{T}'(s) = \kappa(s) \mathbf{\hat{N}}(s). \]

Substituting the function \( s \) in this equation yields

\[ \mathbf{T}'(s) = \kappa(s(t)) \mathbf{\hat{N}}(s(t)) = \kappa(t) N(t) \]

by the definition of \( \kappa \) and \( N \) in the arbitrary-speed case. We know that \( ds/dt = \| \mathbf{\alpha}'(t) \| \) from the definition of arc length

\[ s = \int_0^t \| \mathbf{\alpha}'(u) \| \, du. \]

Denote by \( v = \| \mathbf{\alpha}'(t) \| \) the speed function of \( \mathbf{\alpha} \). Equations (3) and (4) combine to yield

\[ T' = \kappa v N. \]  

(5)

Now let \( \mathbf{\alpha}(t) = (x(t), y(t)) \). Then

\[ T = (x', y')/\| \mathbf{\alpha}'(t) \| = (x', y')/\sqrt{x'^2 + y'^2}, \]

\[ N = (-y', x')/\sqrt{x'^2 + y'^2}. \]

Substituting these terms into (5) yields a formula for evaluating the curvature:

\[
\kappa = \frac{T' \cdot N}{v} = \frac{\left( \frac{(x'', y'')}{(x'^2 + y'^2)} + \frac{d}{dt} \left( \frac{1}{\sqrt{x'^2 + y'^2}} \right) (x', y') \right) \cdot \frac{(-y', x')}{\sqrt{x'^2 + y'^2}}}{\sqrt{x'^2 + y'^2}} \\
= \frac{x' y'' - x'' y'}{(x'^2 + y'^2)^{3/2}}.
\]

We can write the formula simply as

\[ \kappa = \frac{\mathbf{\alpha}' \times \mathbf{\alpha}'''}{\| \mathbf{\alpha}' \|^3}. \]

EXAMPLE 3. Find the curvature of the curve \( \mathbf{\alpha}(t) = (t^3 - t, t^2) \). so we have

\[
\mathbf{\alpha}'(t) = (3t^2 - 1, 2t), \quad \mathbf{\alpha}''(t) = (6t, 2).
\]

Therefore

\[
\kappa = \frac{x' y'' - x'' y'}{(x'^2 + y'^2)^{3/2}} = \frac{(3t^2 - 1) \cdot 2 - 2t \cdot 6t}{\left( (3t^2 - 1)^2 + (2t)^2 \right)^{3/2}} = \frac{6t^2 + 2}{(9t^4 - 2t^2 + 1)^{3/2}}.
\]

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Finally, we derive the formula for computing total curvature over $[a, b]$. Let $\tilde{\alpha}(s)$ be the unit-speed parametrization of $\alpha$, where $s$ is the arc length function. Let $\tilde{a}$ and $\tilde{b}$ be the parameter values such that

$$\tilde{\alpha}(\tilde{a}) = \alpha(a) \quad \text{and} \quad \tilde{\alpha}(\tilde{b}) = \alpha(b).$$

Then the total curvature of $\tilde{\alpha}$ over $[\tilde{a}, \tilde{b}]$ is given by

$$\int_{\tilde{a}}^{\tilde{b}} \kappa(s) \, ds.$$ 

Since $ds/dt = \|\alpha'(t)\|$, we substitute $t$ for $s$ in the above equation and obtain the total curvature formula

$$\Phi(a, b) = \int_{a}^{b} \kappa(t) \|\alpha'(t)\| \, dt.$$ 

References

