Com S 477/577 Problem Solving Techniques for Applied Computer Science

Final Exam

7:30-9:30am
Thursday, Dec 18, 2003

Name: ________________________________

ID (4-digit): ___ ___ ___ ___
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1. **32 pts** Short Questions

(a) **6 pts** The polynomial $x^6 - 2x^4 + 5x^3 - 2x^2 - 1$ has

- ________ positive real zeros,
- ________ negative real zeros,
- ________ complex zeros.

(b) **6 pts** Characterize the SVD solution to the linear system $Ax = b$, where $A$ is an $n \times n$ square matrix. [Note that $A$ may be singular and $b$ may not be in the column space of $A$.]

(c) **2 pts** To observe a signal of frequency $f$, we should sample it at frequency

- ________.
(d) [4 pts] Write the Jacobian of the vector function \( f = (\cos \theta \sin \phi, -\sin \theta \cos \phi)^T \) in variables \( \theta \) and \( \phi \).

(e) [6 pts] Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a twice continuously differentiable function. Give the first and second order necessary conditions on a relative minimum \( \mathbf{x}^* \).

(f) [3 pts] Let \( p(x) \) be the polynomial that interpolates a trigonometric function \( f(x) \) at distinct points \( x_0, x_1, \ldots, x_n \). The leading coefficient of \( p(x) \) is

\[ \text{________________________.} \]
(g) [5 pts] Let \( \mathbf{a}(t) \) be a curve in \( \mathbb{R}^3 \). Denote by \( v, \kappa, \tau, T, N, B \) its speed, curvature, torsion, tangent, principal normal, and binormal, respectively. Give the Frenet formulas that describe the curve.

2. [15 pts] Approximation

Let \( p_0(x), \ldots, p_k(x) \) be an orthogonal sequence of polynomials under the inner product

\[
\langle g, h \rangle = \int_a^b g(x)h(x) \, dx.
\]

Here \( \deg(p_i) = i \) for \( 0 \leq i \leq k \). Consider a function \( f(x) \) defined on \( [a, b] \).

(a) [6 pts] Give the least-squares approximation of \( f(x) \) by any polynomial of degrees at most \( k \).
(b) [9 pts] Prove that your answer in (a) is indeed optimal in the least squares sense.
3. [15 pts] Linear programming

A small firm specializes in making five types of spare automobile parts. Each part is first cast from iron in the casting shop and then sent to the finishing shop where holes are drilled, surfaces are turned, and edges are ground. The required worker-hours (per 100 units) for each of the parts of the two shops are shown below:

<table>
<thead>
<tr>
<th>part</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<tbody>
<tr>
<td>casting</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>finishing</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>1</td>
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</tbody>
</table>

The profits from the parts are $30, $20, $40, $25, and $10 (per 100 units), respectively. The capacities of the casting and finishing shops over the next month are 700 and 1000 worker-hours, respectively.

(a) [6 pts] Formulate the problem of determining the quantities of each spare part to be made during the month so as to maximize profit.

(b) [9 pts] Use the simplex method to solve the formulated problem.
4. [10 pts] *Nonlinear Programming*

Find the point on the plane $ax + by + cz = 1$ that is closest to the origin in $\mathbb{R}^3$. 
5. \[ 8 \text{ pts} \] Algebraic Curves

A lemniscate of Bernoulli is described by the equation

\[(x^2 + y^2)^2 = 2(x^2 - y^2).\]

Find the tangent line and the normal line passing through a point \((a, b)\) on the curve, where \(a \neq 0\) and \(b \neq 0\).
6. [20 pts] Frenet Frame

You are given a curve parametrized as
\[ \mathbf{\alpha}(t) = (3t - t^3, 3t^2, 3t + t^3). \]

(a) [4 pts] What is the speed \( v \) of the curve?

(b) [16 pts] Compute the curve's Frenet apparatus, that is, determine its curvature \( \kappa \), torsion \( \tau \), tangent \( T \), principal normal \( N \), and binormal \( B \).