Com S 477/577 Problem Solving Techniques for Applied Computer Science
Final Exam Practice Problems*

December 8, 2003

1. Short questions.
   (a) Let $f(x_1, x_2, \ldots, x_n)$ be a function from $\mathbb{R}^n$ to $\mathbb{R}$. Give the gradient and Hessian of $f$.
   (b) How many points are used for bracketing at each step in the golden section search for minimizing a nonlinear function $f(x)$? And what relationships should they have?
   (c) Why does the steepest descent method converge slowly to a local minimum?
   (d) A cusp on a curve $\alpha(t)$ at $t = a$ satisfies the condition ____________________.
   (e) A singular point $(a, b)$ on an algebraic curve $f(x, y) = 0$ satisfies the condition ____________________.

2. What is the best uniform approximation by a cubic to the function $\sin \pi x$ over the interval $[-3, 3]$?

3. Calculate the polynomial of degree at most 3 that best approximates $\sin \pi x$ over the interval $[-1, 1]$ in the least-squares sense. Use the first four Legendre polynomials:

   \[ p_0(x) = 1; \]
   \[ p_1(x) = x; \]
   \[ p_2(x) = x^2 - \frac{1}{3}; \]
   \[ p_3(x) = x^3 - \frac{3}{5}x. \]

*These problems are meant to help you prepare for the Final. They do not necessarily reflect the range, quantity, or difficulty of the problems to appear in the exam.
4. You are given an inner product over the interval [0, 1]:
\[ \langle g(x), h(x) \rangle = \int_0^1 g(x) \cdot h(x) \cdot x \, dx. \]
(a) Are the two polynomials below orthogonal under the defined inner product?
\[ p_0(x) = 1 \quad \text{and} \quad p_1(x) = x - \frac{2}{3} \]
(b) Find the orthogonal projection of \( x^2 \) onto the subspace spanned by \( p_0(x) \) and \( p_1(x) \) under the inner product. Is this projection the least-squares approximation of \( x^2 \) over \([0, 1]\) by a linear polynomial?
(c) Use your result from (b) to construct a quadratic polynomial \( p_2(x) \) with leading coefficient 1 that is orthogonal to \( p_0(x) \) and \( p_1(x) \).
(d) Given \( n \) supporting points \((x_i, f_i), i = 1, \ldots, n\), suppose you want to minimize
\[ \sqrt[6]{\sum_{i=1}^{n} (f_i - p(x_i))^6} \]
over all polynomials \( p \) of degree at most 8. What inner product should be used here?

5. Consider the linear program
\[
\begin{align*}
\text{maximize} & \quad x_1 + x_2 + x_3 \\
\text{subject to} & \quad -2x_1 + x_2 + 3x_3 \leq 1; \\
& \quad x_1 - x_2 - 4x_3 = 2; \\
& \quad x_1, x_2, x_3 \geq 0.
\end{align*}
\]
(a) Transform the above linear program into the standard form with an objective function to be minimized subject to equality constraints only.
(b) Use the simplex method to find a basic feasible solution to the standard form.
(c) Does the original linear program have a bounded solution? If yes, find the solution. If no, explain your answer.

6. Consider the linear program
\[
\begin{align*}
\text{maximize} & \quad 3x_1 + x_2 \\
\text{subject to} & \quad x_1 - x_2 \leq -1 \\
& \quad -x_1 - x_2 \leq -3 \\
& \quad 2x_1 + x_2 \leq 4 \\
& \quad x_1, x_2 \geq 0.
\end{align*}
\]
(a) Transform the above linear program into the standard form with an objective function to be minimized subject to equality constraints only.

(b) Use the simplex method to find a basic feasible solution to the standard form. Circle or box the pivot used at each step.

(c) Use your result from (b) to solve the original linear program.

7. A cardboard box for packing quantities of small foam balls is to be manufactured. The top, bottom, and front faces must be of double weight (i.e., two pieces of cardboard). A problem posed is to find the dimensions of such a box that maximize the volume for a given amount of cardboard, equal to 72 square feet.

8. Find a point on the unit sphere \( x^2 + y^2 + z^2 = 1 \) that maximize the objective function

\[
2x + y + 3z.
\]

(a) What are the first-order necessary conditions?

(b) Find \( x, y, z \).

9. Consider the parametric curve \( \alpha(t) = (x(t), y(t)) \) defined on \((0, \infty)\) by

\[
x(t) = \int_0^t \cos u \sqrt{u} \, du, \quad y(t) = \int_0^t \sin u \sqrt{u} \, du.
\]

(a) Find the velocity and speed of the curve.

(b) Obtain its curvature function.

(c) Compute the arc length and total curvature of the curve over the subdomain \([1, 2]\).

(d) Does the curve have any inflections? If so, find them. How about vertices? If so, find them too.
10. Show that the curve \( \alpha(t) = a(\sin(2bt), \sin(bt)) \) is regular and that its tangent is parallel to the \( y \)-axis if and only if \( bt = \pm \frac{\pi}{4} \) or \( \pm \frac{3\pi}{4} \). Find the point where the tangent is parallel to the \( x \)-axis. Determine the curvature at those points where the tangent is parallel to either axis.

11. Compute the curvature \( \kappa \), torsion \( \tau \), unit tangent \( T \), principal normal \( N \), and binormal \( B \) for the following curve:

\[
\alpha(t) = \left( \frac{4}{5} \cos t, 1 - \sin t, -\frac{3}{5} \cos t \right).
\]

12. Given the curve \( \alpha(t) = (2t, t^2, t^3/3) \), compute its Frenet apparatus: curvature \( \kappa \), torsion \( \tau \), unit tangent \( T \), principal normal \( N \), and binormal \( B \).