Com S 477/577 Problem Solving Techniques for Applied Computer Science

Final Exam Practice Problems – Sample Solutions

Tuesday, Dec 16, 2003

1(a)

$$\nabla f = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right),$$

$$\nabla^2 f = \left( \begin{array}{cccc}
\frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\
\frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} 
\end{array} \right).$$

(b) Three points $a < b < c$ such that $f(b) < f(a)$ and $f(b) < f(c)$.

(c) Successive steps in the steepest descent method are along the perpendicular directions, hence often undoing some of their previous progress.

(d) $\alpha'(a) = 0$.

(e) $\nabla f(a, b) = 0$.

2. From a quick sketch of $\sin \pi x$ over the interval $[-3, 3]$ we easily find that this function achieves its largest distance 1 from the $x$-axis (which represents the function $y = 0$) at $6 > 3 + 2$ values $x = -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2},$ and $\frac{5}{2}$. Furthermore, the signs of the difference $\sin \pi x - 0$ alternate at these $x$ values. By Chebyshev’s theorem, the unique best uniform approximation of $\sin \pi x$ by a cubic must be $y = 0$.

3. Choose the inner product $\langle f, g \rangle = \int_{-1}^{1} fg \, dx$. The least-squares approximation of $\sin \pi x$ by a cubic takes the form

$$p(x) = \frac{\langle p_0(x), \sin \pi x \rangle}{\langle p_0(x), p_0(x) \rangle} p_0(x) + \frac{\langle p_1(x), \sin \pi x \rangle}{\langle p_1(x), p_1(x) \rangle} p_1(x) + \frac{\langle p_2(x), \sin \pi x \rangle}{\langle p_2(x), p_2(x) \rangle} p_2(x) + \frac{\langle p_3(x), \sin \pi x \rangle}{\langle p_3(x), p_3(x) \rangle} p_3(x).$$

Let us first calculate the projections of $\sin \pi x$ onto $p_i(x)$, $i = 0, 1, 2, 3$:

$$\langle p_0(x), \sin \pi x \rangle = \int_{-1}^{1} \sin \pi x \, dx$$

$$= -\frac{1}{\pi} \cos \pi x \bigg|_{-1}^{1}$$
\[
\langle p_1(x), \sin \pi x \rangle = \int_{-1}^{1} x \sin \pi x \, dx \\
= x \left( -\frac{1}{\pi} \right) \cos \pi x \bigg|_{-1}^{1} - \int_{-1}^{1} \left( -\frac{1}{\pi} \right) \cos \pi x \, dx \\
= \frac{1}{\pi} + \frac{1}{\pi} - 0 \\
= \frac{2}{\pi};
\]

\[
\langle p_2(x), \sin \pi x \rangle = \int_{-1}^{1} \left( x^2 - \frac{1}{3} \right) \sin \pi x \, dx \\
= \int_{-1}^{1} x^2 \sin \pi x \, dx \\
= x^2 \left( -\frac{1}{\pi} \right) \cos \pi x \bigg|_{-1}^{1} - \int_{-1}^{1} 2x \left( -\frac{1}{\pi} \right) \cos \pi x \, dx \\
= \frac{2}{\pi} \int_{-1}^{1} x \cos \pi x \, dx \\
= \frac{2}{\pi} \left( \frac{1}{\pi} x \sin \pi x \bigg|_{-1}^{1} - \int_{-1}^{1} \frac{1}{\pi} \sin \pi x \, dx \right) \\
= 0;
\]

\[
\langle p_3(x), \sin \pi x \rangle = \int_{-1}^{1} \left( x^3 - \frac{3}{5} x \right) \sin \pi x \, dx \\
= \int_{-1}^{1} x^3 \sin \pi x \, dx - \frac{3}{5} \cdot \frac{2}{\pi} \\
= -\frac{1}{\pi} \cos \pi x \cdot x^3 \bigg|_{-1}^{1} + \int_{-1}^{1} 3x^2 \cdot \frac{1}{\pi} \cos \pi x \, dx - \frac{6}{5\pi} \\
= \frac{2}{\pi} - \frac{6}{5\pi} + \frac{3}{\pi} \left( \frac{1}{\pi} \sin \pi x \cdot x^2 \bigg|_{-1}^{1} - \int_{-1}^{1} \frac{1}{\pi} \sin \pi x \cdot 2x \, dx \right) \\
= \frac{4}{5\pi} - \frac{6}{\pi^2} \int_{-1}^{1} \sin \pi x \cdot x \, dx \\
= \frac{4}{5\pi} - \frac{6}{\pi^2} \cdot \frac{2}{\pi} \\
= \frac{4}{5\pi} - \frac{12}{\pi^3}.
\]

Next, we calculate the squared norms of \( p_1(x) \) and \( p_3(x) \) under the defined inner product:

\[
\langle p_1(x), p_1(x) \rangle = \int_{-1}^{1} x^2 \, dx 
\]
\[
\langle p_3(x), p_3(x) \rangle = \int_{-1}^{1} \left( x^3 - \frac{3}{5} x \right)^2 \, dx \\
= \int_{-1}^{1} \left( x^6 - \frac{6}{5} x^4 + \frac{9}{25} x^2 \right) \, dx \\
= \left. \left( \frac{1}{7} x^7 - \frac{6}{25} x^5 + \frac{3}{25} x^3 \right) \right|_{-1}^{1} \\
= \frac{2}{7} - \frac{12}{25} + \frac{6}{25} \\
= \frac{8}{175}.
\]

Finally, we obtain the least-squares approximation

\[
p(x) = \frac{3}{2} \cdot \frac{2}{\pi} x + \frac{4}{8} \cdot \frac{12}{\pi^4} \left( x^3 - \frac{3}{5} x \right) \\
= \frac{3}{2} \cdot \frac{2}{\pi} x + \left( \frac{35}{2\pi} - \frac{525}{2\pi^3} \right) x^3 \\
= \left( \frac{3 \pi - 21}{2\pi^3} + \frac{315}{2\pi^3} \right) x + \left( \frac{35}{2\pi} - \frac{525}{2\pi^3} \right) x^3.
\]

4(a) We obtain the inner product

\[
\langle p_0, p_1 \rangle = \int_{0}^{1} \left( x - \frac{2}{3} \right) x \, dx \\
= \int_{0}^{1} x^2 - \frac{2}{3} x \, dx \\
= \left. \frac{1}{3} x^3 - \frac{1}{3} x^2 \right|_{0}^{1} \\
= 0.
\]

Hence \( p_0 \) and \( p_1 \) are orthogonal.

(b) To project \( x^2 \) onto the subspace spanned by \( p_0 \) and \( p_1 \), we need to compute the following inner products:

\[
\langle p_0, p_0 \rangle = \int_{0}^{1} x \, dx = \frac{1}{2}, \\
\langle p_1, p_1 \rangle = \int_{0}^{1} \left( x - \frac{2}{3} \right)^2 x \, dx
\]
\[ = \int_0^1 x^3 - \frac{4}{3} x^2 + \frac{4}{9} x \, dx \]
\[ = \frac{1}{36}, \]
\[ \langle x^2, p_0 \rangle = \int_0^1 x^3 \, dx = \frac{1}{4}, \]
\[ \langle x^2, p_1 \rangle = \int_0^1 x^2 \left( x - \frac{2}{3} \right) \, dx \]
\[ = \int_0^1 x^4 - \frac{2}{3} x^3 \, dx \]
\[ = \frac{1}{30}. \]

The orthogonal projection of \( x^2 \) is therefore

\[ \frac{\langle x^2, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0 + \frac{\langle x^2, p_1 \rangle}{\langle p_1, p_1 \rangle} p_1 = \frac{6}{5} x - \frac{3}{10}. \]

No, the projection is not the least-squares approximation because of the weighting function \( w \).

(c) The difference between \( x^2 \) and its orthogonal projection onto the subspace spanned by \( p_0 \) and \( p_1 \) must be orthogonal to both \( p_0 \) and \( p_1 \). This difference,

\[ x^2 - \frac{6}{5} x + \frac{3}{10}, \]

is then the sought quadratic polynomial.

(d) It suffices to minimize

\[ \sum_{i=1}^n (f_i - p(x_i))^6, \]

which should result from some inner product

\[ \langle f - p, f - p \rangle. \]

So the inner product is chosen as

\[ \langle g, h \rangle = \sum_{i=1}^n (g(x_i))^3 (h(x_i))^3. \]

5(a) Convert the original linear program into its standard form after introducing a slack variable \( x_4 \):

minimize \(-x_1 - x_2 - x_3\)
subject to \(-2x_1 + x_2 + 3x_3 + x_4 = 1;\)
\[ x_1 - x_2 - 4x_3 = 2; \]
\[ x_1, x_2, x_3, x_4 \geq 0. \]
(b) In Phase I, we introduce two artificial variables \( x_5 \) and \( x_6 \) and minimize the objective function \( x_5 + x_6 \). The tableau sequence is given below:

\[
\begin{array}{cccccc}
-2 & 1 & 3 & 1 & 1 & 0 & 1 \\
1 & -1 & -4 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1 & 1 & 0 \\
\end{array}
\]

\[
\begin{array}{cccccc}
-2 & 1 & 3 & [1] & 0 & 1 \\
1 & -1 & -4 & 0 & 0 & 1 & 2 \\
1 & 0 & 1 & -1 & 0 & 0 & -3 \\
\end{array}
\]

\[
\begin{array}{cccccc}
-2 & 1 & 3 & 1 & 1 & 0 & 1 \\
[1] & -1 & -4 & 0 & 0 & 1 & 2 \\
-1 & 1 & 4 & 0 & 1 & 0 & -2 \\
\end{array}
\]

\[
\begin{array}{cccccc}
0 & -1 & -5 & 1 & 1 & 2 & 5 \\
1 & -1 & -4 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
\end{array}
\]

So we have found a basic feasible solution \( x_1 = 2, x_4 = 5, x_2 = x_3 = 0 \).

(c) Phase II of the simplex method starts with the tableau

\[
\begin{array}{cccccc}
0 & -1 & -5 & 1 & 5 \\
[1] & -1 & -4 & 0 & 2 \\
-1 & -1 & -1 & 0 & 0 \\
\end{array}
\]

Pivoting once yields

\[
\begin{array}{cccccc}
0 & -1 & -5 & 1 & 5 \\
1 & -1 & -4 & 0 & 2 \\
0 & -2 & -5 & 0 & 2 \\
\end{array}
\]

The second and third columns are the only ones other than the last column that have negative entries in the last row; i.e., only the coefficients of \( x_2 \) and \( x_3 \) are negative in the present cost function. But all other entries in these two columns are negative as well. Therefore we can make the values of \( x_2 \) and \( x_3 \) arbitrarily large without violating the two constraints; in other words, the value of the cost function can be decreased arbitrarily. Hence the original linear program does not have a bounded solution.

6(a) Introduce three slack and surplus variables \( x_3, x_4, x_5 \) and obtain the standard form:

\[
\begin{align*}
\text{minimize} & \quad -3x_1 - x_2 \\
\text{subject to} & \quad -x_1 + x_2 - x_3 = 1 \\
& \quad x_1 + x_2 - x_4 = 3 \\
& \quad 2x_1 + x_2 + x_5 = 4 \\
& \quad x_1, x_2, x_3, x_4, x_5 \geq 0.
\end{align*}
\]
(b) In Phase I, we introduce three artificial variables \(x_5, x_6, x_7\) and minimize the objective function \(x_5 + x_6 + x_7\). The tableau sequence is given below:

\[
\begin{array}{cccccccc}
-1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 3 \\
2 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 4 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{array}
\]

\[
\begin{array}{cccccccc}
-1 & \boxed{1} & -1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 3 \\
2 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 4 \\
\hline
-2 & -3 & 1 & 1 & -1 & 0 & 0 & 0 & -8
\end{array}
\]

\[
\begin{array}{cccccccc}
-1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \\
2 & \boxed{0} & -1 & 0 & -1 & 1 & 0 & 2 \\
3 & 0 & 1 & 0 & 1 & -1 & 0 & 1 & 3 \\
\hline
-5 & 0 & -2 & 1 & -1 & 3 & 0 & 0 & -5
\end{array}
\]

\[
\begin{array}{cccccccc}
1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 3 \\
2 & 0 & 1 & -1 & 0 & -1 & 1 & 0 & 2 \\
1 & 0 & 0 & 1 & \boxed{1} & 0 & -1 & 1 & 1 \\
\hline
-1 & 0 & 0 & -1 & -1 & 1 & 2 & 0 & -1
\end{array}
\]

\[
\begin{array}{cccccccc}
1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 3 \\
2 & 0 & 1 & -1 & 0 & -1 & 1 & 0 & 2 \\
1 & 0 & 0 & 1 & 1 & 0 & -1 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{array}
\]

Hence we have found a basic feasible solution

\[x_1 = 0, \quad x_2 = 3, \quad x_3 = 2, \quad x_4 = 0, \quad x_5 = 1.\]

(c) Phase II of the simplex method starts with the tableau

\[
\begin{array}{cccc}
1 & 1 & 0 & -1 & 0 & 3 \\
2 & 0 & 1 & -1 & 0 & 2 \\
1 & 0 & 0 & 1 & 1 & 1 \\
\hline
-3 & -1 & 0 & 0 & 0 & 0
\end{array}
\]

\[
\begin{array}{cccc}
1 & 1 & 0 & -1 & 0 & 3 \\
2 & 0 & 1 & -1 & 0 & 2 \\
\boxed{1} & 0 & 0 & 1 & 1 & 1 \\
\hline
-2 & 0 & 0 & -1 & 0 & 3
\end{array}
\]

6
\[
\begin{array}{cccc}
0 & 1 & 0 & -2 \\
0 & 0 & 1 & -3 \\
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
\end{array}
\]

So the maximum of the original program is 5 achieved at \( x_1 = 1, x_2 = 2, \) and \( x_3 = 0. \)

7. We first formulate the problem as below:

\[
\begin{aligned}
& \text{max } xyz \\
& \text{subject to } 4xy + 3xz + 2yz = 72
\end{aligned}
\]

At the maximization point, the Lagrangian

\[
f(x, y, z, \lambda) = xyz + \lambda(4xy + 3xz + 2yz - 72)
\]

must have vanishing partial derivatives with respect to \( x, y, z \) and the Lagrange multiplier \( \lambda: \)

\[
\begin{align*}
yz + 4\lambda y + 3\lambda z &= 0, \\
xz + 4\lambda x + 2\lambda z &= 0, \\
xy + 3\lambda x + 2\lambda y &= 0, \\
3xz + 4xy + 2yz &= 72.
\end{align*}
\]

Here equation (4) is same as the original constraint.

Multiply equation (1) with \( x \) and equation (2) with \( y \) and compare the results:

\[
\begin{align*}
xyz + 4\lambda xy + 3\lambda xz &= xyz + 4\lambda xy + 2\lambda yz, \\
3\lambda xz &= 2\lambda yz, \\
x &= \frac{2}{3} y.
\end{align*}
\]

The last step was valid because \( \lambda \neq 0. \) Otherwise, at least two of \( x, y, z \) would be zero according to equations (1), (2), (3), yielding zero volume. Similarly, multiplying (2) with \( y \) and (3) with \( z \) and comparing the results give us

\[
y = \frac{3}{4} z \quad \text{and thus} \quad x = \frac{1}{2} z.
\]

Substitute the above into (4):

\[
72 = 4 \cdot \frac{1}{2} \cdot \frac{3}{4} z^2 + 3 \cdot \frac{1}{2} z^2 + 2 \cdot \frac{3}{4} z^2,
\]

\[
= \frac{9}{2} z^2,
\]

\[
z = 4.
\]

Thus we obtain \( x = 2 \) and \( y = 3. \) And the maximum volume is therefore 24.
8. Define the Lagrangian

\[ F(x, y, z) = 2x + y + 3z + \lambda(x^2 + y^2 + z^2 - 1), \]

where \( \lambda \) is the Lagrange multiplier. At an extremum, all partial derivatives of \( F \) must vanish, yielding

\[
\begin{align*}
2 + 2\lambda x &= 0, \\
1 + 2\lambda y &= 0, \\
3 + 2\lambda z &= 0, \\
x^2 + y^2 + z^2 &= 1.
\end{align*}
\]

(5) (6) (7) (8)

In the above, (4) is just the original constraint. From (5), (6), and (7) we obtain that

\[
x = -\frac{1}{\lambda}, \quad y = -\frac{1}{2\lambda}, \quad z = -\frac{3}{2\lambda}.
\]

This is possible because \( \lambda \neq 0 \). Substituting all the above into (8), we obtain

\[
\frac{1}{\lambda^2} + \frac{1}{4\lambda^2} + \frac{9}{4\lambda^2} = 1,
\]

\[
\lambda = \pm \sqrt{\frac{7}{2}}.
\]

Clearly, we need only keep \( \lambda = -\sqrt{\frac{7}{2}} \), which yields

\[
x = \frac{2}{\sqrt{14}}, \quad y = \frac{1}{\sqrt{14}}, \quad z = \frac{3}{\sqrt{14}}.
\]

So the maximum is

\[
2 \cdot \frac{2}{\sqrt{14}} + \frac{1}{\sqrt{14}} + 3 \cdot \frac{3}{\sqrt{14}} = \frac{14}{\sqrt{14}} = \sqrt{14}.
\]

9(a) We first obtain the derivatives:

\[
x'(t) = \frac{\cos t}{\sqrt{t}}, \\
y'(t) = \frac{\sin t}{\sqrt{t}}.
\]

The speed and velocity of the curve are

\[
\alpha'(t) = \frac{(\cos t, \sin t)}{\sqrt{t}},
\]

\[
\|\alpha'(t)\| = \frac{1}{\sqrt{t}}.
\]
(b) We perform the following calculations:

\[
\begin{align*}
    x''(t) &= -\frac{\sin t}{\sqrt{t}} - \frac{\cos t}{2(\sqrt{t})^3}, \\
    y''(t) &= \frac{\cos t}{\sqrt{t}} - \frac{\sin t}{2(\sqrt{t})^3}, \\
    x'y'' - x''y' &= \frac{\cos^2 t}{t} - \frac{\sin t \cos t}{2t^2} + \frac{\sin^2 t}{t} + \frac{\sin t \cos t}{2t^2} \\
    &= \frac{1}{t}.
\end{align*}
\]

Therefore the curvature is

\[
    \kappa = \frac{x'y'' - x''y'}{\|\alpha'(t)\|^3} = \sqrt{t}.
\]

(c) The arc length over \([0, 1]\) is

\[
    \int_1^2 \frac{1}{\sqrt{t}} \, dt = 2\sqrt{t} \bigg|_1^2 = 2\sqrt{2} - 2.
\]

And the total curvature over the same subdomain is

\[
    \int_1^2 \kappa \|\alpha'(t)\| \, dt = \int_1^2 \sqrt{t} \cdot \frac{1}{\sqrt{t}} \, dt = t \bigg|_1^2 = 1.
\]

(d) Since \(\kappa = \sqrt{t} > 0\) over \((0, \infty)\), the curve does not have an inflection over \((0, \infty)\). Since \(\kappa' = \frac{1}{2\sqrt{t}} > 0\), it does not have a vertex either.

10. We obtain the first two derivatives of \(\alpha(t)\):

\[
\begin{align*}
    \alpha'(t) &= ab\left(2 \cos(2bt), \cos(bt)\right), \\
    \alpha''(t) &= -ab^2\left(4 \sin(2bt), \sin(bt)\right).
\end{align*}
\]

From the above we see that the tangent \(\alpha'(t)\) is parallel to the \(y\)-axis if and only if \(2bt = \frac{\pi}{2}\) mod \(2\pi\) or \(\frac{3\pi}{2}\) mod \(2\pi\), i.e., \(bt = \pm\frac{\pi}{4}\) or \(\pm\frac{3\pi}{4}\). At a point with tangent parallel to the \(x\)-axis, \(\cos(bt) = 0\). So \(bt = \frac{\pi}{2}\) or \(\frac{3\pi}{2}\).

The speed of the curve is therefore

\[
\begin{align*}
    &ab\sqrt{4\cos^2(2bt) + \cos^2(bt)} \\
    &= ab\sqrt{4\left(2 \cos^2(bt) - 1\right)^2 + \cos^2(bt)} \\
    &= ab\sqrt{16 \cos^4(bt) - 15 \cos^2(bt) + 4} \\
    &= ab\sqrt{4\cos^2(bt) - \frac{15}{8}} + \frac{31}{64} \\
    &\geq \frac{\sqrt{31}}{8} ab \\
    &> 0.
\end{align*}
\]


Hence the curve is regular.
Next, we calculate the curvature function
\[ \kappa(t) = \frac{\alpha' \times \alpha''}{\|\alpha'\|^3} \]
\[ = \frac{-a^2 b^3 \left(2 \sin(bt) \cos(2bt) - 4 \sin(2bt) \cos(bt)\right)}{a^3 b^3 \left(\sqrt{4 \cos^2(2bt) + \cos^2(bt)}\right)^3} \]
\[ = \frac{-2 \sin(bt) \cos(2bt) - 4 \sin(2bt) \cos(bt)}{a \left(\sqrt{4 \cos^2(2bt) + \cos^2(bt)}\right)^3}. \]

Now we can evaluate the curvature
\[ \kappa = \begin{cases} \pm \frac{1}{4a} & \text{if the tangent is parallel to the x-axis.} \\ \pm \frac{8}{a} & \text{if the tangent is parallel to the y-axis.} \end{cases} \]

11. We calculate the first three derivatives of \( \alpha(t) \):
\[ \alpha'(t) = \left(-\frac{4}{5} \sin t, - \cos t, \frac{3}{5} \sin t\right), \]
\[ \alpha''(t) = \left(-\frac{4}{5} \cos t, \sin t, \frac{3}{5} \cos t\right), \]
\[ \alpha'''(t) = \left(\frac{4}{5} \sin t, \cos t, -\frac{3}{5} \sin t\right). \]

Since both \( \alpha'(t) \) and \( \alpha''(t) \) are unit vectors, we immediately have
\[ T = \alpha'(t), \]
\[ N = \alpha''(t), \]
\[ \kappa = 1. \]

And the binormal vector can be obtained as follows:
\[ B = T \times N = \alpha'(t) \times \alpha''(t) = \left(-\frac{3}{5}, 0, -\frac{4}{5}\right). \]

That the binormal vector is a constant implies that the torsion
\[ \tau = 0. \]

12. We carry out a sequence of steps to compute the Frenet apparatus:
\[ \alpha'(t) = (2, 2t, t^2), \]
\[
\begin{align*}
\alpha''(t) &= (0, 2, 2t), \\
\alpha'''(t) &= (0, 0, 2), \\
\|\alpha'\| &= \sqrt{4 + 4t^2 + t^4} \\
&= t^2 + 2, \\
T &= \left(\frac{2, 2t, t^2}{t^2 + 2}\right), \\
\alpha' \times \alpha'' &= (2t^2, -4t, 4), \\
\|\alpha' \times \alpha''\| &= \sqrt{4t^4 + 16t^2 + 16} \\
&= 2(t^2 + 2), \\
B &= \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|} \\
&= (t^2, -2t, 2), \\
N &= B \times T \\
&= \frac{(-2t^3 - 4t, 4 - t^4, 2t^3 + 4t)}{(t^2 + 2)^2}, \\
&= \left(\frac{-2t, 2 - t^2, 2t}{t^2 + 2}\right), \\
\kappa &= \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3} \\
&= \frac{2}{(t^2 + 2)^2}, \\
\tau &= \frac{(\alpha' \times \alpha'') \cdot \alpha'''}{\|\alpha' \times \alpha''\|^2} \\
&= \frac{8}{4(t^2 + 2)^2} \\
&= \frac{2}{(t^2 + 2)^2}.
\end{align*}
\]