Home Work 1
Com S 531
Due Feb 10: 11am

Notation:

\( \mathbb{N} \) is the set of Natural numbers. \( \mathbb{R} \) is the set of Real numbers. \( \mathbb{E} \) is the set of all even numbers. \( \mathbb{Z} \) is the set of all integers. Finally, \( \mathbb{Q}^+ \) is the set of positive rational numbers. Note that every member of this set is bigger than 0.

1. Show that if \( A \) and \( B \) are countable, then \( A \cup B \) is countable.

2. Show that for every infinite set \( S \) there exists an infinite countable set \( S' \) such that \( S' \subseteq S \).

3. A function \( f : S \to \mathbb{N} \) is called inf-to-one if for every \( n \in \mathbb{N} \) \( |f^{-1}(n)| = |\mathbb{N}| \). Suppose that there is a inf-to-one function from a set \( S \) to \( \mathbb{N} \). What can you say about the cardinality of \( S \)? Is it finite? countable or uncountable? Prove your answer.

4. Show that there exist infinite sets \( N_1, N_2, \cdots \) such that
   
   \begin{itemize}
   \item For every \( i \neq j \), \( N_i \cap N_j = \emptyset \).
   \item \( \bigcup_{i=1}^{\infty} N_i = \mathbb{N} \).
   \end{itemize}

5. Let \( S \) be an infinite set that is not countable so that \( \mathbb{N} \subseteq S \). Show that \( |S - \mathbb{N}| = |S| \).

6. Let \( S \) be the set of all functions from \( \mathbb{Z} \) to \( \mathbb{E} \). Show that \( S \) is not countable. Your proof must do a direct diagonalization over \( S \).

7. Let \( S \) be the set of all functions from \( \mathbb{N} \) to \( \mathbb{Q}^+ \) so that for every function \( f \) in \( S \)
   
   \[ \sum_{i=1}^{\infty} f(i) \leq 1 \]

   Show that \( S \) is not countable by doing a diagonalization argument over \( S \).

8. Show that the set of all onto functions from \( \mathbb{N} \) to \( \mathbb{N} \) is not countable.

9. Let \( S \) be an uncountable set. Is there a reasonable way to describe every element of \( S \) (in English)? What about in French? Is there a (unknown) language in which we can reasonably describe every element of \( S \)? Well, I haven’t defined the meaning of “reasonably describing an element”. Part of your job is to understand this notion of “reasonably describing an element”.

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Guidelines:

• You are allowed to discuss with your classmates, while doing your homework. However, I strongly suggest that you think about the problems on your own before discussing.

• Definition of classmates: Students who are taking CS 531 in Spring 14.

• However, You should write the final solutions alone, without consulting your classmates. Your writing should demonstrate that you understand the proofs completely. If I suspect that you wrote the proofs without understanding, I may ask you to explain the proofs to me in person. In such scenarios, failure to explain proofs will be taken as evidence of academic dishonesty.

• For each problem, you should acknowledge the students with whom you discussed. This will not affect your grade. Failure to acknowledge is considered academic dishonesty, and it will affect your grade.

• Any student found guilty of academic dishonesty will receive “F” in the course.

• When proofs are required, make them both clear and rigorous. Do not hand wave. Even when proofs are not required, you should justify your answers and explain your work.

• Late homeworks are not accepted.