Approximation Algorithms for Weighted Vertex Cover

CS 511

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Weighted Vertex Cover: Problem Definition

**Input:** An undirected graph $G = (V, E)$ with vertex weights $w_i \geq 0$.

**Problem:** Find a minimum-weight subset of nodes $S$ such that every $e \in E$ is incident to at least one vertex in $S$. 
Weighted Vertex Cover: Some Facts

- WVC is NP-hard.
- WVC can be 2-approximated.
  - Proved next.
- A 2-approximation algorithm for WVC does not provide any sort of approximation guarantee for maximum-weight independent set.
Weighted Vertex Cover: IP Formulation

\[
\begin{align*}
\text{minimize} & \quad \sum_{i \in V} w_i x_i \\
\text{subject to} & \quad x_i + x_j \geq 1 \quad \text{for every edge } (i, j) \in E \\
& \quad x_i \in \{0, 1\} \quad \text{for every vertex } i \in V
\end{align*}
\] (1)

Observation

- Any feasible solution \( x \) to (1) yields a cover \( S = \{i \in V : x_i = 1\} \).
- If \( x^* \) is optimal solution to (1), then \( S^* = \{i \in V : x_i^* = 1\} \) is a minimum weight vertex cover.
Weighted Vertex Cover: LP Relaxation

minimize $\sum_{i \in V} w_i x_i$
subject to $x_i + x_j \geq 1$ for every edge $(i, j) \in E$
$x_i \geq 0$ for every vertex $i \in V$

Observation

The optimum value of LP relaxation (2) is at most equal to the optimum value of integer program (1), because the LP has fewer constraints.
The LP-Rounding Algorithm

1. Compute the optimum solution $x^*$ to LP relaxation (2).
2. Let $S = \{ i \in V : x^*_i \geq 1/2 \}$.
3. Return $S$. 
Theorem

The LP-Rounding Algorithm is a 2-approximation algorithm for MWVC.

Proof.

1. **S is a vertex cover.** Consider an edge \((i, j) \in E\). Since \(x_i^* + x_j^* \geq 1\), either \(x_i^* \geq 1/2\) or \(x_j^* \geq 1/2\) \(\Rightarrow\) \((i, j)\) is covered.

2. **If \(S^*\) is an optimum vertex cover, then \(w(S) \leq 2w(S^*)\).**

\[
\begin{align*}
    w(S^*) &\geq \sum_{i=1}^{n} w_i x_i^* \quad \text{since LP is a relaxation of ILP} \\
    &\geq \sum_{i \in S} w_i x_i^* \quad \text{since } S \subseteq \{1, \ldots, n\} \\
    &\geq \frac{1}{2} \sum_{i \in S} w_i \quad \text{since } x_i^* \geq 1/2 \text{ for all } i \in S \\
    &= \frac{1}{2} w(S)
\end{align*}
\]
Weighted Vertex Cover: Dual LP

maximize \[ \sum_{e \in E} y_e \]
subject to \[ \sum_{e=(i,j) \in E} y_e \leq w_i \quad \text{for every node } i \in V \] \[ y_e \geq 0 \quad \text{for every edge } e \in E \]
Intuition for Duality

- Edge $e$ pays price $y_e \geq 0$ to be covered.
- Goal: Collect as much money as possible from the edges.
- **Fair price condition:** For every $i \in V$,

$$
\sum_{e=(i,j) \in E} y_e \leq w_i.
$$
The Dual Gives a Lower Bound

Lemma (Fairness Lemma)

Let \((y_1, y_2, \ldots, y_{|E|})\) be any feasible solution to the dual LP and \(S^*\) be a minimum-weight vertex cover. Then,

\[
\sum_{e \in E} y_e \leq w(S^*).
\]

Proof.

Let \(z^*_D, z^*_P, \) and \(z^*_IP\) be the optimal objective values of the dual LP, the primal LP, and the primal ILP for WVC. Then,

\[
\sum_{e \in E} y_e \leq z^*_D \quad \text{dual optimality} \quad z^*_D = z^*_P \quad \text{strong duality} \quad z^*_P \leq z^*_IP = w(S^*). \quad \text{integrality}
\]
Using Duality

• Any dual solution \( y \) gives a lower bound on the optimal solution to WVC — don’t need to solve dual to optimality.
• However, \( y \) should be easy to convert into a vertex cover \( S \).
• Further, \( S \) should not be too far from optimum.
• We can find such a \( y \) by a simple and fast method, without using an LP solver.
The Pricing Method

Definition

Vertex $i$ is tight if $\sum_{e=(i,j)\in E} y_e = w_i$.

Pricing-Method($G, w$)

\[
\begin{align*}
\text{for each } e &\in E \\
y_e &\equal 0 \\
\text{while } \text{there is an edge } (i,j) &\text{ such that neither } i \text{ nor } j \text{ are tight} \\
&\text{select such an edge } e \\
&\text{increase } y_e \text{ as much as possible while preserving dual feasibility} \\
S &\equal \{i \in V : i \text{ is tight}\} \\
\text{return } S
\end{align*}
\]
The Pricing Method: Example

Source: Kleinberg & Tardos, *Algorithm Design* (Fig. 11.8).
Theorem

The Pricing Method is a 2-approximation algorithm for MWVC.
Theorem

The Pricing Method is a 2-approximation algorithm for MWVC.

Proof.

1. Running time is polynomial.
   
   *Reason:* At least one new node becomes tight after each iteration of while loop.

2. The set $S$ returned is a vertex cover.
   
   *Reason:* At termination, for each edge $e = (u, v)$, at least one of $u$ and $v$ is tight $\implies$ at least one of $u$ and $v$ is in $S$. 
Theorem

The Pricing Method is a 2-approximation algorithm for MWVC.

Proof.

Let $S^*$ be an optimum vertex cover. Then $w(S) \leq 2w(S^*)$.

Reason:

$$w(S) = \sum_{i \in S} w_i$$

$$= \sum_{i \in S} \sum_{e=(i,j)} y_e$$

since the nodes in $S$ are tight

$$\leq \sum_{i \in V} \sum_{e=(i,j)} y_e$$

since $S \subseteq V$ and $y_e \geq 0$ for all $e$

$$= 2 \sum_{e \in E} y_e$$

because each edge is counted twice

$$\leq 2w(S^*)$$

by the Fairness Lemma