This closed-book, closed-notes two-hour test consists of 6 questions. The number of points for each problem is indicated on the next page.

- Read all questions carefully before starting.
- Work on the problems that seem easiest first.
- Attempt to solve all problems.
- Show your work, but also remember that we prefer concise answers.
- Write all your answers clearly on the space provided in the exam paper. If you need additional paper, please ask us.
- If you do not understand a problem, please ask us for clarification.
- Clearly state any simplifying assumptions you make in solving a problem.
- When asked to describe algorithm, you are expected to argue its correctness and analyze its running time.

Name: ____________________________________________

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## Score

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1 NP-Completeness (16 points)

As we saw in class, the following problem is NP-complete.

**HAMILTONIAN CYCLE**

**Input:** Undirected graph \( G = (V, E) \).

**Question:** Does \( G \) have a cycle that visits each vertex exactly once?

Consider **HAMILTONIAN CYCLE** restricted to graphs in which every vertex has degree at most 2. Call this problem **HAMILTONIAN CYCLE-2**.

(a) (4 points) Prove that **HAMILTONIAN CYCLE-2** is in NP.
(b) (6 points) What is wrong with the following proof of NP-completeness for HAMILTONIAN CYCLE-2?

We know that the HAMILTONIAN CYCLE problem in general graphs is NP-complete, so it is enough to present a reduction from HAMILTONIAN CYCLE-2 to HAMILTONIAN CYCLE. Given a graph $G$ with vertices of degree at most 2, the reduction leaves the graph unchanged: clearly the output of the reduction is a possible input for the HAMILTONIAN CYCLE problem. Furthermore, the answer to both problems is identical. This proves the correctness of the reduction and, therefore, the NP-completeness of HAMILTONIAN CYCLE-2.
(c) (6 points) Show that HAMILTONIAN CYCLE-2 can be solved in polynomial time.
2 Decision, Search, and Optimization (16 points)

As you recall, the subset sum problem is defined as follows.

**SUBSET SUM**

**Input:** A set of integers \( S = \{w_1, w_2, \ldots, w_n\} \) and an integer \( W \).

**Question:** Does there exist a subset of \( S \) that adds up to exactly \( W \)?

(a) (8 points) Suppose you have a procedure solves that solves SUBSET SUM in polynomial time. That is, this procedure takes an instance \( \langle S, W \rangle \) of SUBSET SUM and returns “yes” if there is a subset of \( S \) that adds up to \( W \), and returns “no” otherwise. Show that you can use this procedure to develop a polynomial-time algorithm that returns a subset of \( S \) that adds up to \( W \), if such a subset exists, or reports that no such subset exists otherwise.
(b) (8 points) Consider the following optimization version of SUBSET SUM:

**Max Subset Sum**

**Input:** A set of positive integers $S = \{w_1, w_2, \ldots, w_n\}$ and a positive integer $W$.

**Goal:** Find a subset of $S$ whose sum is as large as possible, without exceeding $W$.

Show that Max Subset Sum is solvable in polynomial time if and only if Subset Sum is.
3 Proving NP-Completeness by Generalization (18 points)

For each of the problems below, prove that it is NP-complete by (i) arguing why it is in NP and (ii) stating which NP-complete problem it generalizes and how.

(a) (6 points) MIN UNSAT: Given a CNF formula $\varphi$ and an integer $g \geq 0$, determine whether there exists a truth assignment where at most $g$ clauses of $\varphi$ are not satisfied.
(b) (6 points) **SPARSE SUBGRAPH**: Given a graph $G$ and two integers $a \geq 0$ and $b \geq 0$, determine if there is a subset of at least $a$ vertices of $G$ such that there are at most $b$ edges between them.
(c) (6 points) SUBGRAPH ISOMORPHISM: Given as input two undirected graphs $G$ and $H$, determine whether $G$ is a subgraph of $H$ (that is, whether by deleting certain vertices and edges of $H$ we obtain a graph that is, up to renaming of vertices, identical to $G$).
4 Degree-Constrained Spanning Trees (16 points)

A spanning tree of a connected, undirected graph \( G = (V, E) \) is a tree \( T \) whose vertex set is \( V \) and whose edge set is a subset of \( E \) (note that, by definition, \( T \) must be connected). Now consider the following problem.

\[ k \text{-SPANNING TREE} \]

**Input:** An undirected graph \( G = (V, E) \).

**Output:** A spanning tree of \( G \) in which each node has degree at most \( k \), if such a tree exists.

Note that we place no restriction on the degrees of the nodes in \( G \).

(a) (4 points) Show that, for \( k \geq 2 \), \( k \text{-SPANNING TREE} \) is in NP.
(b) (12 points) Show that, for $k \geq 2$, $k$-SPANNING TREE is NP-complete. (*Hint:* What happens when $k = 2$?)
5 The Knapsack Problem (16 points)

Consider the following problem.

KNAPSACK

Input: A collection of \( n \) items, where item \( i \) has an integer weight \( w_i \) and an integer value \( v_i \), along with two integers, \( W \) and \( V \).

Question: Does there exist a subset of the items whose total weight is at most \( W \) and whose total value is at least \( V \)?

(a) (4 points) Prove that KNAPSACK \( \in \) NP.
(b) (12 points) Show that KNAPSACK is NP-complete. (*Hint:* Use reduction from SUBSET SUM.)
6 Partitioning into Communities (18 points)

An interaction matrix for a set $P$ of $n$ people is a zero-one matrix $C = [c_{ij}]$, where

$$c_{ij} = \begin{cases} 
1 & \text{if person } i \text{ is known to interact with person } j \text{ on a social basis,} \\
0 & \text{otherwise.}
\end{cases}$$

Note that $C$ is symmetric; i.e., $c_{ij} = c_{ji}$. Assume that $c_{ii} = 1$ for $i = 1, \ldots, n$.

Let us define a community to be a subset $A$ of $P$ such that every two people in $A$ interact. Consider the following problem

**PARTITIONING INTO COMMUNITIES**

**Input:** An interaction matrix $C$ for a set $P$ of $n$ people and an integer $K$.

**Question:** Can $P$ be partitioned into at most $K$ disjoint communities?

(a) (4 points) Show that PARTITIONING INTO COMMUNITIES is in NP.
(b) (14 points) Show that PARTITIONING INTO COMMUNITIES is NP-complete. (Hint: Consider the case where $K = 3$.)