Approximation Algorithms for Weighted Vertex Cover

CS 511

Iowa State University

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Weighted Vertex Cover: Problem Definition

**Input:** An undirected graph $G = (V, E)$ with vertex weights $w_i \geq 0$.

**Problem:** Find a minimum-weight subset of nodes $S$ such that every $e \in E$ is incident to at least one vertex in $S$. 
Weighted Vertex Cover: Some Facts

- WVC is NP-hard.
- WVC can be 2-approximated.
  ▶ Proved next.
- A 2-approximation algorithm for WVC does not provide any sort of approximation guarantee for maximum-weight independent set.
Weighted Vertex Cover: IP Formulation

minimize \[ \sum_{i \in V} w_i x_i \]
subject to \[ x_i + x_j \geq 1 \] for every edge \((i, j) \in E\) \[ x_i \in \{0, 1\} \] for every vertex \(i \in V\)

Observation

- Any feasible solution \(x\) to (1) yields a cover \(S = \{i \in V : x_i = 1\}\).
- If \(x^*\) is optimal solution to (1), then \(S^* = \{i \in V : x_i^* = 1\}\) is a minimum weight vertex cover.
Weighted Vertex Cover: LP Relaxation

minimize \( \sum_{i \in V} w_i x_i \)
subject to
\[ x_i + x_j \geq 1 \text{ for every edge } (i, j) \in E \]
\[ x_i \geq 0 \text{ for every vertex } i \in V \]  \hspace{1cm} (2)

Observation
The optimum value of LP relaxation (2) is at most equal to the optimum value of integer program (1), because the LP has fewer constraints.
The LP-Rounding Algorithm

1. Compute the optimum solution $x^*$ to LP relaxation (2).
2. Let $S = \{i \in V : x_i^* \geq 1/2\}$.
3. Return $S$. 
Theorem

The LP-Rounding Algorithm is a 2-approximation algorithm for MWVC.

Proof.

1. *S* is a vertex cover. Consider an edge \((i,j) \in E\). Since 
   \(x_i^* + x_j^* \geq 1\), either 
   \(x_i^* \geq 1/2\) or 
   \(x_j^* \geq 1/2 \Rightarrow (i,j)\) is covered.

2. If \(S^*\) is an optimum vertex cover, then \(w(S) \leq 2w(S^*)\).

\[
\begin{align*}
  w(S^*) &= \sum_{i \in S^*} w_i \\
  &\geq \sum_{i \in S} w_i x_i^* \quad \text{Since LP is a relaxation of ILP} \\
  &\geq \frac{1}{2} \sum_{i \in S} w_i \quad \text{Since } x_i^* \geq 1/2 \\
  &= \frac{1}{2} w(S)
\end{align*}
\]
The Pricing Method

**Intuition**

Edge $e$ pays price $p_e \geq 0$ to be covered by vertex $i$.

**Fairness Condition**

$$\forall i \in V, \quad \sum_{e=(i,j)} p_e \leq w_i$$

If equality holds, $i$ is said to be paid for or tight.
The Pricing Method

Lemma (Fairness Lemma.)

For any vertex cover $S$ and any fair prices $p_e$,

$$\sum_{e \in E} p_e \leq w(S).$$

Proof.

$$\sum_{e \in E} p_e \leq \sum_{i \in S} \sum_{e=(i,j)} p_e \leq \sum_{i \in S} w_i = w(S).$$
The Pricing Method

\[ \text{Pricing-Method}(G, w) \]

\textbf{for each} \( e \in E \)
\begin{align*}
\text{do} \quad & p_e \leftarrow 0 \\
\text{while} \ (\text{there is an edge } (i, j) \text{ such that neither } i \text{ nor } j \text{ are tight}) \\
\text{do} \quad & \text{select such an edge } e \\
& \text{increase } p_e \text{ as much as possible without violating fairness}
\end{align*}

\( S \leftarrow \text{set of all tight nodes} \)

\textbf{return} \( S \)
The Pricing Method: Example

Source: Kleinberg & Tardos, *Algorithm Design* (Fig. 11.8).
Theorem

The Pricing Method is a 2-approximation algorithm for MWVC.
Theorem

The Pricing Method is a 2-approximation algorithm for MWVC.

Proof.

1. Running time is polynomial.
   *Reason:* At least one new node becomes tight after each iteration of while loop.

2. The set $S$ returned is a vertex cover.
   *Reason:* At termination, for each edge $e = (u, v)$, at least one of $u$ and $v$ is tight $\implies$ at least one of $u$ and $v$ is in $S$. 
Theorem

The Pricing Method is a 2-approximation algorithm for MWVC.

Proof.

Let $S^*$ be an optimum vertex cover. Then $w(S) \leq 2w(S^*)$.

Reason:

$$w(S) = \sum_{i \in S} w_i$$

$= \sum_{i \in S} \sum_{e=(i,j)} p_e$ \hspace{1cm} \text{(nodes in $S$ are tight)}

$\leq \sum_{i \in V} \sum_{e=(i,j)} p_e$ \hspace{1cm} \text{($S \subseteq V$ and $p_e \geq 0$ for all $e$)}

$= 2 \sum_{e \in E} p_e$ \hspace{1cm} \text{(each edge is counted twice)}

$\leq 2w(S^*)$ \hspace{1cm} \text{(by fairness lemma)}