Rearrangement of Analytical C to an O(N²) formula

To transform your analytical generation of C from an O(N^3) to an O(N^2) operation you need to know some summation identities and to modify them appropriately.

So to do this lets start from the beginning. We are multiplying a matrix A by B to generate C. Let’s assume that:

\[ A(M, N) \times B(N, L) = C(M, L) \]  (1)

We have chosen an arbitrary analytical definition of A and B:

\[ A(i, j) = a \times i + b \times j + c \]  (2)

\[ B(i, j) = d \times i + e \times j + f \]  (3)

where \( i, j \) are the row and column indecies, respectively and \( a, b, c, d, e, f \) are arbitrary compile time constants. The range of \( i \) or \( j \) is from zero to the rank minus one in that dimension of the matrix.

Based on the definition of matrix multiplication we know that each element of C is computed by the formula:

\[ C_{i,j} = \sum_{k=0}^{N-1} A_{i,k} \times B_{k,j} \]  (4)

By substituting the definitions of A (Equation 2) and B (Equation 3) into Equation 4 we get:

\[ C_{i,j} = \sum_{k=0}^{N-1} (a \times i + b \times k + c) \times (d \times k + e \times j + f) \]  (5)

Now by expanding all terms in Equation 5 and collecting those terms with respect to those that multiply powers of k we get:

\[ C_{i,j} = \sum_{k=0}^{N-1} bdk^2 + \sum_{k=0}^{N-1} [d(ai + c) + b(ej + f)]k + \sum_{k=0}^{N-1} [(ai + c)(ej + f)] \]  (6)

We want to identify each of the summations as individual terms so we redefine Equation 6 as three summation terms

\[ C_{i,j} = T_{i,j}^1 + T_{i,j}^2 + T_{i,j}^3 \]  (7)

where each term is:

\[ T_{i,j}^1 = \sum_{k=0}^{N-1} bdk^2 \]  (8)
\[ T_{i,j}^2 = \sum_{k=0}^{N-1} [d(ai + c) + b(ej + f)]k \]  \hspace{1cm} (9)

\[ T_{i,j}^3 = \sum_{k=0}^{N-1} [(ai + c)(ej + f)] \]  \hspace{1cm} (10)

Now we know from various mathematical tables that the three identity sums we need are:

\[ \sum_{k=1}^{n} 1 = n \]  \hspace{1cm} (11)

\[ \sum_{k=1}^{n} i = \frac{n(n + 1)}{2} \]  \hspace{1cm} (12)

\[ \sum_{k=1}^{n} i^2 = \frac{n(2n + 1)(n + 1)}{6} \]  \hspace{1cm} (13)

Note that the range of the sum in each Equation is from 1 to \( n \) for these formulas. Let’s look at each term in Equation 6. The first term Equation 8 has a multiplicative constant \( bd \) that can be pulled out of the summation to give:

\[ T_{i,j}^1 = \sum_{k=0}^{N-1} bd k^2 = (bd) \sum_{k=0}^{N-1} k^2 \]  \hspace{1cm} (14)

Now if we let \( n = N - 1 \) in Equation 13 and we notice that \( i^2 = 0 \) when \( i = 0 \) we get:

\[ \sum_{k=0}^{N-1} k^2 = \frac{N(2N - 1)(N - 1)}{6} \]  \hspace{1cm} (15)

Using this result in Equation 14 we get:

\[ T_{i,j}^1 = bd \frac{N(2N - 1)(N - 1)}{6} \]  \hspace{1cm} (16)

Note that there is no formal dependence on \( i \) or \( j \) in this Equation!

The second term, Equation 9, has a multiplicative constant that can be pulled out of the summation as well to give:

\[ T_{i,j}^2 = \sum_{k=0}^{N-1} [d(ai + c) + b(ej + f)]k = [d(ai + c) + b(ej + f)] \sum_{k=0}^{N-1} k \]  \hspace{1cm} (17)

Now if we let \( n = N - 1 \) in Equation 12 and we notice that when \( i = 0 \) we have no additional contribution we get:

\[ \sum_{k=0}^{N-1} i = \frac{N(N - 1)}{2} \]  \hspace{1cm} (18)
Using this result in Equation 17 we get:

\[
T_{i,j}^2 = [d(ai + c) + b(ej + f)] \frac{N(N - 1)}{2}
\]

(19)

The third term, Equation 10 only has a multiplicative constant and no \( k \) dependence in the summation. This gives:

\[
T_{i,j}^3 = [(ai + c)(ej + f)] \sum_{k=0}^{N-1} 1
\]

(20)

Now close examination of Equation 11 shows that the summation turns into a multiplier that is determined by the range of the starting and ending values of the summation. Based on this you can show that:

\[
\sum_{k=1}^{n} 1 = \sum_{k=0}^{n-1} 1 = n
\]

(21)

or more generally:

\[
\sum_{k=startvalue}^{endvalue} 1 = endvalue - startvalue + 1
\]

(22)

Using this result in Equation 20 we get:

\[
T_{i,j}^3 = [(ai + c)(ej + f)]N
\]

(23)

Now substituting Equations 16, 19, and 23 into Equation 7 we get:

\[
C_{i,j} = bd \left[ \frac{N(2N - 1)(N - 1)}{6} \right] \\
+ [d(ai + c) + b(ej + f)] \left[ \frac{N(N - 1)}{2} \right] \\
+ [(ai + c)(ej + f)]N
\]

(24)

Which has no \( k \) dependence at all since all \( k \) dependences have been “summed” out of the equation. Using Equation 24 for each element of \( C \) you can develop an \( \mathcal{O}(N^2) \) function for the analytical generation of the \( C \) matrix.